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THE MATHEMATICS TEACHER

VOLUME XXI **MARCH, 1928** NUMBER 3

A CHAPTER ON THE ÆSTHETICS OF THE QUADRATIC

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1. THE FUNDAMENTAL CHARACTERS

It may not have occurred to the student that mathematics is one of the fine arts, and exhibits the same characters as those we find in poetry, music, painting, and sculpture. The material or medium of mathematics is of a more subtle stuff than that used in the arts mentioned, for it is dealing with purely ideal or non-material objects. But it takes little reflection on the problem to see that we may find in every part of mathematics instances of the qualities that determine a work of art. The object of this chapter is to bring some of these to the notice of the student.

2. DESIGN

Everywhere in art we find design. A mere chaos of objects, of colors, of sounds, would not make a work of art. It is not meant that everything must have an uninteresting, monotonous, stiff regularity, for there is design in artistic irregularity. But design means order. It also means composition. Order relates to the way things are arranged or put together, while composition relates to the general pattern. For instance, in a piece of music we find order in the melody and the working out of the theme, and composition in the succession of phrases and the harmonizing. The whole makes a comprehensive design, as in a symphony. So, too, in a poem we have an order in the succession of ideas and what they suggest, and we have composition in the development of the poem, the whole constituting one design.

In the case of the quadratic $Ax^2 + Bx + C$ we find order in the fact that there is a term not containing x, one with x, and one with x^2. This means that whatever number x may be,

such a form made up of these three terms is a quadratic. But it also means that whatever expression in a parenthesis may be used in place of x, we still have a quadratic. For instance,

$$3(y+2)^2 - 5(y+2) + 16, \quad z - e + \sqrt{z-e} + 8, \quad x^8 + 4x^4 - 15,$$

are quadratics. That we may recognize a given form as a quadratic it is evidently necessary that we should be able to recognize a square when we see it. In the case of numbers this is easy for the smaller numbers, and writing out the squares of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, we have 1, 4, 9, 16, 25, 36, 49, 64, 81, 100. Looking over this list we see there are none that end with 2, 3, 7, 8. Any number with these digits as terminal figure can not be a perfect square. There is evidently then a design also underlying the squares of integers, and the ambitious student will be able to follow up this topic in books on the Theory of Numbers. Again, if we square a monomial, as x^n, we get x^{2n}; consequently, expressions that are to be considered to be squares must be written with even exponents. When we square a binomial, as $A + B$, we have $A^2 + B^2 + 2AB$, and for a trinomial we have $(A + B + C)^2 = A^2 + B^2 + C^2 + 2AB + 2AC + 2BC$; and by considering polynomials in general, we see that the design of a square of a polynomial will always have the squares of each of the terms, and added to these the double products of each pair of terms. Therefore, whenever we can so order an expression that we have first some coefficient multiplied by a square of a parenthesis, following this a coefficient multiplied by the parenthesis itself, and finally a term independent of the parenthesis, we may be sure that the expression under consideration is a quadratic.

Exercises, to be Ordered and Written as Quadratics.—

1. $16x - 4x^2 + 32.$
2. $4x^2 - 8 + 6x.$
3. $x^2 + 2xy + y^2 + 6x + 6y - 8.$
4. $9x^2 + 12xy + 4y^2 + 15x + 10y + 6.$
5. $x + 5 - \sqrt{x+5} - 6.$
6. $x^4 - 14x^2 + 40.$
7. $(x^4 - 6x^3 + 9x^2) + 5x^2 - 15x - 14.$
8. $x^4 + 2x^3 + x^2 - 3x^2 - 3x - 108.$
9. $x^4 + 2x^3 - 15x^2 - 16x + 12.$
10. $x^2 + 4x + 3 + \sqrt{9x^2 + 36x + 27} - 4.$

Besides order in the quadratic we find composition. This is shown in two ways, the first being a pattern common to other polynomials besides quadratics, the second belonging to the quadratic alone. The first pattern of composition of the quadratic is shown in the fact that it always consists of two linear factors. The second pattern is shown in the fact that every quadratic can be written as the difference of two squares.

It is easy to see that if we multiply together $x - r$ and $x - s$ we shall have a quadratic, $x^2 - (r + s)x + rs$. What we have to see further is that any quadratic can be written in this form or pattern. The problem can be stated also thus:

Let the quadratic be $Ax^2 + Bx + C$: to write it in the form $A(x - r)(x - s)$. It is clear at once that we must have

$$r + s = -\frac{B}{A},$$
$$rs = \frac{C}{A}.$$

If we can solve these two equations and find r and s, we shall be able to factor the quadratic. This may often be done by mere inspection. For instance, let the quadratic be $x^2 - 14x + 40$. Then $r + s = 14$ and $rs = 40$. Considering the factors of 40, which are 1 and 40, 2 and 20, 4 and 10, 5 and 8, we see that 4 and 10 will make the correct sum, so that the factors are $(x - 4)(x - 10)$. Also if the quadratic is $15x^2 - 38x + 24$, it is clear we must have factors of 15 attached to the x in each factor, and of 24 to take the places of the second term in each factor. The factors of 15 are 1 and 15, 3 and 5. The factors of 24 are 1 and 24, 2 and 12, 3 and 8, 4 and 6. We must so arrange them that the cross-products of the factors will add up to 38. A little trial will show that

$$\begin{array}{c} 3x - 4 \\ 5x - 6 \\ \hline 15x^2 - 38x + 24 \end{array}$$

In most cases, however, the matter of finding the factors is not so simple. For instance, let the quadratic be $x^2 - 8x + 3$. We have here

$$r + s = 8,$$
$$rs = 3.$$

It is clear that no integers will satisfy these equations. If we try to solve them by finding r from the first, $r = 8 - s$, then substituting in the second, we have

$$8s - s^2 = 3.$$

Arranging this in order, we have $s^2 - 8s + 3 = 0$. But this is simply the original quadratic over, and we have gone around in a circle. We must take a different line for the solution. The solution of the quadratic dates back many centuries, in fact it was known to Diophantos in the fourth century A.D. Whatever the form it may take, it is contained essentially in the following. If we square $r + s$, we have

$$r^2 + 2rs + s^2 = 64.$$

Then from the second equation

$$4rs = 12.$$

Subtracting, we have

$$r^2 - 2rs + s^2 = 52.$$

But this is the square of $r - s$, so we may write

$$(r - s)^2 = 52.$$

From this we see that

$$r - s = +\sqrt{52} \text{ or } -\sqrt{52}.$$

Or, in a somewhat simpler form for some situations,

$$r - s = +2\sqrt{13} \text{ or } -2\sqrt{13}.$$

Then since

$$r + s = 8,$$

by adding and subtracting, and dividing by 2,

$$r = 4 + \sqrt{13}, \qquad s = 4 - \sqrt{13}.$$

The factored pattern then is

$$(x - 4 - \sqrt{13})(x - 4 + \sqrt{13}).$$

These forms show us why we could not get the factors by inspection. They also show us that if we are dealing with problems where irrational numbers are not possible the quadratic will not have factors. That is, the quadratic in some cases is *irreducible* in the *domain of rational numbers.* Irrational numbers were

created in order to relieve such a situation. This is analogous to the creation of semi-tones in music in order to express the full forms of music. It is in fact a characteristic of all art that it introduces ideal objects in order to complete its world.

If now we carry out this same work for the quadratic $Ax^2 + Bx + C$, we have

$$r + s = -\frac{B}{A},$$

$$rs = \frac{C}{A},$$

$$(r - s)^2 = \frac{B^2 - 4AC}{A^2}.$$

Taking only the positive sign of the square root, which means merely that we are choosing r to be the greater of the two numbers r and s, we have

$$r - s = \frac{\sqrt{B^2 - 4AC}}{A}.$$

The expression $B^2 - 4AC$ is so important we give it a special symbol, Δ, and a name, the *discriminant* of the quadratic. If we now add and subtract as before, we have

$$r = \frac{-B + \sqrt{\Delta}}{2A},$$

$$s = \frac{-B - \sqrt{\Delta}}{2A}.$$

This is called the formula solution of the quadratic. It should be noticed that the half sum of r and s furnishes the first term in the fraction, and the half difference the second term. The factor pattern of the quadratic is now completely exhibited. It shows that the domain for the expression depends upon the number Δ. If it is a perfect square and positive, the factors are rational. If it is not a square, or if it is negative, the factors are called irrational. In case it is negative we have to create numbers whose square will be negative in order to carry out the work. Such numbers are called imaginary, though they are not any more imaginary than is any number. They should have been called ideal numbers. The ideal square root of -1 is usually represented by i, so that we should write

$$\sqrt{-9} = 3i, \qquad \text{and} \qquad \sqrt{-12} = i2\sqrt{3}.$$

The powers of i repeat their values, thus, $i^2 = -1$, $i^3 = -i$, $i^4 = +1$, $i^5 = i$, $i^6 = -1$, etc.

The pattern for the quadratic which is peculiar to it is the difference of two squares. To discover this we start again with the quadratic

$$Ax^2 + Bx + C,$$

which we write in the form

$$A\left(x^2 + \frac{B}{A}x + \frac{C}{A}\right).$$

Now remembering that the expression $x^2 + 2mx + m^2$ is a square, we see that if we find half of $\frac{B}{A}$ or $\frac{B}{2A}$ it will be m, and if we square it, then

$$x^2 + \frac{B}{A}x + \frac{B^2}{4A^2}$$

is a square. We can write the quadratic now in the form

$$A\left(x^2 + \frac{B}{A}x + \frac{B^2}{4A^2} - \frac{B^2}{4A^2} + \frac{C}{A}\right).$$

The terms $-\frac{B^2}{4A^2} + \frac{C}{A}$ may be reduced to one denominator, giving

$$-\frac{B^2 - 4AC}{4A^2} = -\frac{\Delta}{4A^2}.$$

We have then finally for this pattern of the quadratic

$$A\left[\left(x + \frac{B}{2A}\right)^2 - \left(\frac{\sqrt{\Delta}}{2A}\right)^2\right].$$

This form shows that any quadratic can be written as the difference of two squares if we can consider a domain such that Δ is a square. This will make it necessary to choose as domain the field of all rational numbers and all complex numbers. This field will include, of course, numbers which are not the result of purely algebraic operations, so it is larger than it really need be, but if we let A, B, and C be any numbers out of this domain, then it would be necessary to have the entire domain in order to write the quadratic according to this pattern.

As an instance, let the quadratic be $5x^2 + 7x + 11$. We have $\Delta = -171$, and we can write $\sqrt{\Delta} = i3\sqrt{19}$. Hence the quad-

ratic takes the form

$$5\left[\left(x+\frac{7}{10}\right)^2-\left(\frac{i3\sqrt{19}}{10}\right)^2\right].$$

Exercises.—Put into the form of the difference of two squares:

1. $x^2 - 8x + 15$.	6. $2x^2 - 5x - 12$.
2. $x^2 + 6x - 40$.	7. $5x^2 + 18x - 8$.
3. $x^2 - 7x - 78$.	8. $3x^2 - 7x - 12$.
4. $x^2 + 5x - 13$.	9. $4x^2 + 6x + 8$.
5. $x^2 + 5x + 13$.	10. $15x^2 - 24x - 16$.

3. Symmetry

Another marked characteristic of the quadratic is the symmetry that underlies its structure and its applications. In the expressions given above for the sum and the product of the roots, r, s,

$$r + s = -\frac{B}{A},$$

$$rs = \frac{C}{A}.$$

It is evident that if we interchange r and s the equations are not changed. They are symmetric in r and s. So also in the expression

$$(r - s)^2 = \frac{\Delta}{A^2}$$

it is clear that we may also interchange r and s, since the square of a minus is a plus, or $(r - s)^2 = (s - r)^2$. If we take the square root, $r - s$ is not symmetric. We can easily construct other symmetric expressions for r and s and in every case we would be able to find values for them in terms of A, B, C, rationally. For instance

$$r^2 + s^2 = \frac{B^2 - 2AC}{A^2},$$

$$\frac{r}{s} + \frac{s}{r} = \frac{B^2 - 2AC}{AC},$$

$$r^3 + s^3 = \frac{3ABC - B^3}{A^3}.$$

If we consider the factors $x - r$ and $x - s$, we have two

symmetric expressions

$$A(x-r)(x-s) = Ax^2 + Bx + C = q.$$
$$A[(x-r)+(x-s)] = 2Ax - A(r+s) = 2Ax + B = q'.$$

These two expressions are, therefore, derivable at once from the quadratic without knowing r and s, that is, without finding the factors themselves. The second may be written $2A\left(x - \frac{-B}{2A}\right)\cdot$ Now in any factor such as $x - y$, if we give x the value of y we have zero as the value of the factor, and if a factor of a product is zero the whole product is zero. Consequently, to find a value of x which will make a given expression equal to zero, all we need to do is to find a value which will make one of the factors of the expression equal to zero. In other words, the values of x which make a given expression equal to zero are the values which make the various linear factors of the expression zero. These are the roots of the expression. (If the expression is not integral, but has a divisor which has factors containing x, the values of x in these divisors which make them equal to zero give the poles of the fraction.) Above we have r and s for the roots of the quadratic, and for the second of the two symmetric expressions above the root is clearly $\frac{-B}{2A}$ or $\frac{r+s}{2}\cdot$ The root of the second expression is then half the sum of the roots of the quadratic, and may be said to be half way between them. We may also add a third symmetric expression to the above two:

$$A^2[(x-r)-(x-s)]^2 = A^2(r-s)^2 = \Delta.$$

This connects the symmetric expressions of the factors to the symmetric expressions of the roots. We might, of course, construct from these other symmetric expressions, as

$$\begin{aligned} A^2(x-r)^2 + A^2(x-s)^2 &= 2A^2x^2 + 2ABx + B^2 - 2AC \\ &= 2Aq + \Delta, \end{aligned}$$

where q stands for the quadratic itself.

$$\frac{x-r}{x-s} + \frac{x-s}{x-r} = \frac{2A^2x^2 + 2ABx + B^2 - 2AC}{A^2x^2 + ABx + AC} = \frac{2Aq + \Delta}{Aq},$$

$$\begin{aligned} A^3(x-r)^3 + A^3(x-s)^3 &= (2Ax + B)(3A^2x^2 + 3ABx + B^2 - AC) \\ &= q'(3Aq + \Delta). \end{aligned}$$

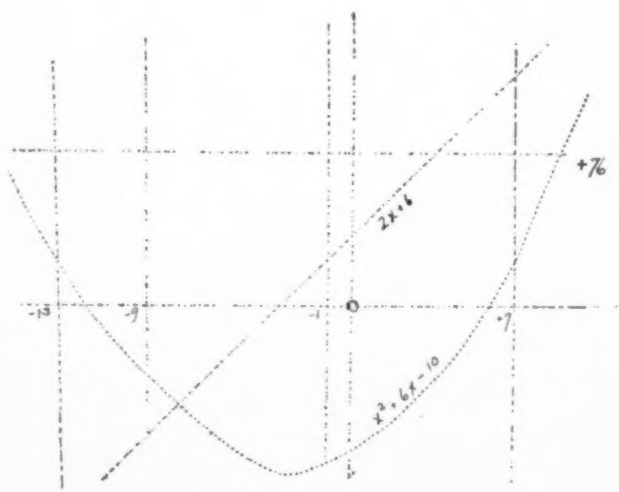
In fact any symmetric expression in $(x - r)$ and $(x - s)$ can be expressed in terms of the coefficients A, B, C, and q, q'. The elementary set of symmetric expressions

$$q = Ax^2 + Bx + C,$$
$$q' = 2Ax + B,$$
$$q_2 = B^2 - 4AC,$$

are called Sturm's functions for the quadratic. There are corresponding extensions for the cubic, the quartic, and for any rational integral polynomial in x.

The significance of Sturm's functions becomes clear if we make a graph for them on one diagram. For example, consider the set

$$q = x^2 + 6x - 10,$$
$$q' = 2x + 6,$$
$$q_2 = 76.$$



If we look at the vertical line through the point $x = -13$, we see that it crosses the graph of the quadratic above the x-axis, the straight line $y = 2x + 6$ below the axis, and the constant line above. This may be indicated definitely by the signs $+ - +$. If we look at the other vertical lines through -9, -1, and $+7$, we can arrange the whole set of signs in the table

	-13	-9	-1	$+7$
$x^2 + 6x - 10$	$+$	$-$	$-$	$+$
$2x + 6$	$-$	$-$	$+$	$+$
76	$+$	$+$	$+$	$+$

This table shows in the first column of signs 2 variations in the signs, in the second column 1 variation, in the third column, in spite of the change in the middle sign, 1 variation, and in the fourth column 0 variations. It is evident that whenever there is a change in sign in the first line, that is, wherever we have passed a crossing of the graph of q, there will be a loss of one variation, but a change in sign in the middle line produces no change in the number of variations. This is a case of an important theorem due to Sturm, for which he was awarded a prize by the French Academy of Sciences.

Examples.—Study the changes in signs of the functions for the quadratics

1. $x^2 - 4x - 7$.
2. $x^2 + 4x - 7$.
3. $x^2 - 4x + 7$.
4. $x^2 + 4x + 7$.
5. $x^2 - 4x + 4$.
6. $x^2 + 6x + 9$.

Symmetry is shown in two forms or phases, balance and value. Balance consists in the arrangement of a design with regard to some central point, or central line. Value has to do with the relative intensity of the effect or importance of the various parts of the design. In color, values are changed by making the colors lighter or darker. In drawing, balance is obtained by making the drawings larger or smaller, more or less detailed or distinct. In the quadratic we find that the balance consists in the fact that the roots may be expressed in the forms

$$p + q \quad \text{and} \quad p - q.$$

It will be noticed that the derived function $2Ax + B$ crosses the x-axis half way between the roots and so furnishes the central point for the quadratic. This point is given by the number p. We may also notice that in the array of signs in the table the very definite system of variations fades out towards the right into a

system of no variations, however far we go in the value of x. It should be noted also by the student that the "quadratic" implies not only the form $Ax^2 + Bx + C$, but all the other forms derivable from this and the forms $2Ax + B$ and $B^2 - 4AC$, which are more or less hidden in the structure that may properly be said to belong to the quadratic.

4. Rhythm

Rhythm in some form is a feature of all art. It means repetitions and cadences. Repetitions are more or less regular and somewhat monotonous, cadences are irregular and varied, but with an underlying rhythm. This is shown very easily to be true of the quadratic.

Suppose we undertake to find the approximate root of the quadratic $x^2 = 3x + 1$. We may write this, after dividing by x,

$$x = 3 + \frac{1}{x}.$$

If now in this we substitute on the right the value of x as given, we have

$$x = 3 + \cfrac{1}{3 + \cfrac{1}{x}}.$$

This process may be continued indefinitely, and as the relative effect on the value of the final fraction is smaller and smaller the farther down the fraction is, it may, for an approximation, be cut off after several terms are written. For instance, we would have these approximations to the root of this equation:

$$x = 3, \quad x = 3 + \frac{1}{3}, \quad x = 3 + \cfrac{1}{3 + \cfrac{1}{3}}, \quad x = 3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3}}},$$

etc. Whether these values "define" the root is not an algebraic question.

An expression of this kind is called a continued fraction. It is usually written on a single line, thus:

$$3 + \frac{1}{3+}\,\frac{1}{3+}\,\frac{1}{3+}\,\frac{1}{3+}\,\frac{1}{3+}\,\frac{1}{3}\cdots.$$

In this form we clearly have a recurrence or repetition. In fact it can be shown that any repeating (periodic) continued fraction must represent the root of some quadratic. In a system of writing fractions on such a scale as a decimal scale (or any other base than ten equally well) a repeating set of figures always represents a rational number. For instance the decimal

$$.\dot{5}6789\dot{2}$$

represents the fraction

$$\frac{567892}{999999}.$$

The theorem that a repeating continued fraction represents a quadratic irrationality is an extension of the theorem of arithmetic.

We may also represent the solution of the quadratic by

$$\tfrac{1}{2}(3 \pm \sqrt{13}).$$

Now $\sqrt{13}$ may be written out as an approximation in a series that runs thus:

$$3\left(1 + \frac{2}{9} - \frac{2}{81} + \frac{4}{729} - \frac{10}{6561} + \cdots\right).$$

In this we have a certain cadence given by the law of the series, the relative effects of the successive terms decreasing as we get farther out.

But, further, there is a more profound rhythm in the quadratic, for we find associated with the quadratic another quadratic which is of the form $z^2 = 13$. This is a resolvent Galois equation for the quadratic. A Galois resolvent equation represents rhythm in a high degree, for it is an equation such that any one of its roots may be expressed in rational form in terms of any other one, and if the various roots are all expressed in terms of a selected one, then the forms of these expressions when applied to any other of the roots will also give the entire system. These forms or operations by which we may find all the roots in terms of any one root of the Galois resolvent equation are such that if any two are applied in succession to one of the roots the result could have been produced by a third operation of the set, and this operation would be independent of the root chosen. This will be exemplified.

In the case of the quadratic above, the resolvent mentioned, $z^2 = 13$, has two roots, and the operation of passing from one to the other is indicated by the statement

$$z_1 = -z_2.$$

There is the quadratic itself, however, also as a resolvent equation, for since the product of the roots is -1, we may indicate either in terms of the other by the equation

$$z_1 = \frac{-1}{z_2}.$$

In this equation z_1 and z_2, of course, do not mean the same as in the preceding. This equation tells us that either root is the negative reciprocal of the other.

Galois was studying the problem of how to express the roots of any equation and in this study brought to light the structure of equations. He found that this was intimately connected with an accompanying resolvent equation, and he was the first to discover the way in which the solutions could be found and expressed.

5. Harmony

Harmony means the consistency of a work of art in itself. Of course there may be and frequently are contrasts introduced that serve to emphasize the essential harmony, but these are, after all, a part of a deeper harmony, a more profound kind of harmony. Harmony is shown, of course, in the whole of mathematics in its general consistency, the contrasts perhaps being shown in the exceptions, in what are called singular cases. We see harmony exhibited in the quadratic in the expression of the quadratic as a graph. The various positions the graph takes for real roots, complex roots, rational or irrational roots, equal roots, together with the Sturm functions as graphs, exhibit not only a harmony between the geometric expression but show harmonies in the algebraic expression. For instance, if the graph is slid along from right to left without being raised or lowered, it is evident that the chord cut off on the x-axis remains unchanged or invariant. In fact this chord is the difference of the roots and may be expressed in terms of the discriminant Δ. At the same time the whole system of Sturm moves along with the graph of the quadratic so that these functions have an invariant

relation to the quadratic itself. Since essentially harmony is a quality due to invariancy, we may see the connection that exists when we leave the difference of the roots invariant.

6. Unity

Unity is that quality of a work of art which may be expressed mathematically by the terms coherence and closure. We find both exemplified in the theory of the quadratic. The theory, of course, is coherent in that when it is fully worked out it is seen to form a set of properties of a definite structure, which belong together, that is, cohere. The properties of the roots, the conditions under which they may be equal, may vanish, may approach infinite values, may be real or complex; the structure of the coefficients in terms of the roots and of functions of the roots in terms of the coefficients; the invariant properties and, to go further, the relations that may exist between two quadratics, or three quadratics;—all indicate a high degree of coherence. Further, the theory is a closed theory, inasmuch as it may be considered a unit, and when fully treated is a finished piece of investigation. This does not mean that the quadratic may not be considered in reference to a more comprehensive theory, for, of course, it belongs to the general theory of algebraic equations, but it is a unit in itself. Every part of mathematics is, of course, part of the grand unity of all mathematics.

If the student will consider these ideas carefully he will find the

Æsthetics of the Quadratic

ON THE NATURE OF ALGEBRAIC LANGUAGE

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Among the practical aims of mathematical instruction recommended by the National Committee we find the following: "An understanding of the language of algebra and the ability to use the language intelligently in the expression of simple quantitative relations; appreciation of the significance of formulas and ability to use this language intelligently." The ability to understand a language and use it as a means of expression is indeed very important, and in the educational process specific provisions have been made for its development. The person who is equipped with this ability is in a position to use it as a study tool, and the more abilities one has acquired in understanding different languages the better prepared is he to engage in extensive studies effectively. But is the language of algebra really a language, and is the understanding and intelligent use of it analogous to the understanding and use of an ordinary language?

We are told that the science of mathematics has a language of its own, and that the marvelous growth of mathematics as an exact science is due to the preciseness of its language. The more a science becomes exact the more it uses the language of mathematics for the expression of its laws and relationships. If we consider the study of mathematics as consisting mainly of the study of the sizes, shapes, properties of entities, and of any quantitative relations between them that are determined either through observation and measurement, or through the process of reasoning, then the symbolic representation of these properties and relationships is indeed the language of mathematics. Now this symbolic representation is not by any means peculiar to algebra for all the branches of mathematics use representative symbols. Then just what is meant by the language of algebra? Is it identical with the language of mathematics, or is it a "dialect" of it?

A little reflection on the nature of the symbolic representa-

tion will enable us to answer these questions intelligently. Instead of working with entities themselves the mathematician has invented a method of working with certain symbols which are representatives of the groups of entities; the operations are performed on these representative symbols and the results interpreted in terms of the entities which they represented. This method enables one to know the properties and characteristics of the group by studying those of its representatives, and to determine the laws which the members of the group obey by observing the law or laws which govern their representatives. Let us illustrate. By measurement, the student finds that the area of a rectangle having the dimensions 4 and 5 units respectively is 20 square units, and that a second rectangle whose dimensions are 4 and 6 units respectively has for its area 24 square units. Yet this knowledge of the relation between the areas and the dimensions of the two special rectangles does not justify the conclusion that the whole group of rectangles obeys the same empirical law. Nor will the work with 20 or more rectangles furnish him with the general law applicable to the whole group. The study of specific cases simply furnishes information which is useful in drawing inferences, sometimes correctly, as to the nature of the general law.

That a certain law or rule holds for 500 specific cases is not a reliable criterion that the law will hold for the 501st case. Of course, many scientific experiments are not carried that far but the experimenter is ready to draw conclusions from the results of only a few experiments. How then determine the general law? For this purpose, referring to our illustration, a representative rectangle is chosen, called "any" rectangle, and the laws which this "any" rectangle obeys are the laws which the group of all rectangles must obey, because the selected rectangle is the representative of the group of rectangles and stands for any member of that group. The dimensions and the area of this representative rectangle are called general numbers, that is they are representatives of the whole group of ordinary arithmetical numbers. Then if the relation $a = bh$, where a, b, and h represent the area and the two dimensions of the general rectangle, is true, any special rectangle having for its dimensions any ordinary arithmetical numbers whatsoever must obey the same law.

This illustration has furnished us with two distinct principles which will enable us to answer the questions raised above concerning the language of algebra. In the first place these general numbers, the representatives of the ordinary arithmetical numbers, are the algebraic expressions. One is tempted to call them algebraic numbers, for that is what they are when the term arithmetical numbers is used to denote ordinary numbers, but the terminology "algebraic member" is reserved by the mathematician for another use in higher mathematics, so we must be contented with the term "algebraic expression." These are the elements with which elementary algebra deals. They are representative symbols of certain groups of elements, whether the group is the field of positive integers, of positive and negative integers, of rational numbers, the real field, or the complex field depends upon the particular algebra under consideration. The important thing to note here is that any notion of representative symbols, or general numbers, is a part of the algebraic language.

In the second place an expression of any law or relationship between these general numbers is algebraic per se. These general numbers may represent any entities or elements whatsoever with which the study of mathematics is concerned. It is in this sense of generality and symbolic representation that we are justified to call algebraic language the language of mathematics.

Now consider the practical aim concerning this language which is recommended by the National Committee. The recommendation specifies the ability of understanding and of intelligent use of this language in the expression of simple quantitative relationships (simple for secondary school mathematics) as signified by the use of formulas. The understanding of formulas which express quantitative relationships, in view of the second principle discussed above, must be interpreted as an understanding of the language of algebra. But this understanding is based, as pointed out above, upon an understanding of the nature of the symbols in terms of which the relationships are expressed. A formula involving equality between certain symbols, such as for example $v = at + v_0$, is meaningless unless the nature of the symbols v, a, t, and v_0 is understood. The fact that the student can manipulate the equation $15 = 2x + 3$ but balks when the equation takes the form $v = at + v_0$ shows clearly that the gen-

eral numbers v, a, and v_0 are meaningless to him, *i.e.*, though he is able to perform algebraic processes he does not understand the language of algebra. To understand this language it is necessary to understand the functional character of the formula, and the ability to understand the latter depends upon a knowledge of the true meaning of the literal coefficients. The manipulation of equations with arithmetical coefficients and constants, though very desirable, does not by itself insure a grasp of the function concept, which is the intelligent use of the algebraic language recommended by the National Committee.

The functional character of an algebraic expression depends entirely upon the interpretation of the meaning of its letters or symbols. If the meaning is that of general or representative numbers then it is obvious that for any set of arithmetical values the expression will have a unique value, and that this arithmetical value of the expression changes with and is dependent upon the set of values of the involved letters. This property of the algebraic expression should be distinctly made clear by the process of evaluation which is treated in all elementary courses in algebra, but evaluation is not often utilized to emphasize functionality. The student is asked, for example, to evaluate many different algebraic expressions for a single set of values of the literal numbers, instead of requiring him to find different values for the same expression for different sets of values of the literal numbers. The latter procedure not only furnishes the desired amount of practice in the process of evaluation, but also brings out clearly the functional character of the algebraic expression; whereas by the former procedure the manipulation of the process of evaluation alone receives the emphasis.

Now the meaning given the literal numbers in the textbooks, when these symbols are first introduced, is not always that of the general numbers. Textbooks in Junior High School Mathematics, and in First Year Algebra use letters in the following sense:

1. Used as labels. For example, a for line $\overset{a}{\overline{\qquad\qquad}}$, m for $\angle AOB$, etc.

2. Used as abbreviations for words. For example, l for length, w for width, and a for area; p for principal, r for rate, t for time, and i for interest.

3. Used for unknown quantities, where the unknown quantity is denoted by x, ?, or by ⋯.

4. Used to denote magnitudes. For example, a for a line segment a units in length, m for an angle m units (degrees) in size.

With the exception of (4), the first three methods fail to give the reader an intelligent attitude toward the use of literal numbers. It may be argued that when an author makes the statement (1) in his text, "using lines a and b as units construct lines whose lengths are $2a + 3b$," he implies that lines a and b are a and b units (centimeters, or inches) in length respectively. Perhaps he does, but it is not clear from the nature of the statement. The above statement is equivalent to, "using centimeters and inches as units construct a line whose length is 2 cm. + 3 in.," for clearly the author wishes the lines (segments) a and b to be used as units of measure. Had the symbols a and b been used previously in the text to denote general or variable quantities, then the meaning of functionality might be attached to the author's statement, but this intended meaning has neither been explained nor illustrated before. They have been used, rather, as labels to designate sets of arithmetical exercises, and are introduced without further explanation to designate lines.

The second procedure, namely, the use of letters as abbreviations for words also may be interpreted to mean that the letters used are perfectly general literal numbers with algebraic meaning. But it is well known that some educators advocate the use of words in explaining mathematical operations and relationships rather than symbols and where symbols are actually used they are meant to be only abbreviations for the words. One author, after stating the relation between interest, principal, rate, and time in words, writes it symbolically $I = P \times R \times T$. Not a word is said about the nature of this symbolic representation which is introduced for the first time. Obviously it is desired that the letters be interpreted only as abbreviations of the words interest, principal, rate, and time, and the multiplication sign for the word multiply. In this formula and the formula $a = l \times w$, where a, l, and w are abbreviations for the words area, length, and width, the emphasis is placed upon the relation of equality in the particular case and not upon the nature of the literal numbers. Give the pupil who has just learned to inter-

pret the formula $a = lw$ an equation of the form $y = xz$ he is puzzled, and justly so, for x, y, and z are not abbreviations for any words with which he is familiar. Later he will be asked to forget that the letters are abbreviations for words, and will be retaught their true meaning.

Incidentally, we wish to point out a procedure in the use of letters which is unnecessarily confusing. There is no reason why small letters could not be uniformly used in elementary mathematics to represent magnitudes, since capital letters are used to designate points, without magnitude.

The third procedure may be convenient in the solution of arithmetical problems, but as a training in the use of algebraic language is very poor pedagogy. The reader will, certainly, not grasp the functionality of an algebraic expression by seeing a question mark imprinted in the face of every literal number that he encounters. Give him the equation $2x + 3 = 7$, he is quite at home for "two times the unknown increased by three gives seven" is a nice little puzzle (?), and he delights (?) in seeking the unknown. But change the equation to $ax + b = c$, he is ready to quit, for "unknown times unknown increased by unknown gives unknown" is a hard nut to crack. The equation, which is an algebraic statement, is meaningless to him if he is to interpret it in terms of the "unknown." True, later on he will be taught how to read and interpret such an equation, but that will follow only as a consequence of learning the true significance of the coefficients a, b, and c. Why not then teach him this meaning of the literal numbers at the outset? For certainly the notion of the general number is not any more difficult than that of hunting a particular one among the group of arithmetical numbers. The point is that with an understanding of the nature of algebraic symbols and expressions not only he is able to catch the significance of the relation $ax + b = c$, but also he will see the equation $2x + 3 = 7$ in its true light, as a special case of the former for the set of values $a = 2$, $b = 3$, and $c = 7$. Furthermore, the algebraic expression $2x + 3$ being a general number may represent any arithmetical number such as the number 7, provided a suitable value is chosen for x. Now the question is asked "for what value of x will the expression $2x + 3$ represent the number 7?." If unknowns are to be sought let them be seen as special or particular values of the general numbers but these

general numbers must not be lost sight of in the hunt for the unknown.

In the fourth procedure the literal numbers are introduced as variable quantities, and the functionality of the algebraic expression may follow as a logical consequence of the method of presentation. In the first place, by letting the literal numbers denote the measure of any line or any angle then the line and the angle are variable quantities, and the letters represent general numbers. In the second place, in a statement like "let the line segment be *a* units in length," if the segment is interpreted to mean a definite line segment with definite length, then the notion of variation is again involved if we consider the fact that by changing the unit of measure the number *a* may be made to represent any arithmetical number whatsoever, whereas if the unit of measure is specified then *a* has a particular value. The interpretation of the literal numbers as designating any line or any angle is to be preferred for it embodies the idea of functionality which is the basic principle of algebra, and it emphasizes the important notion of symbolic representation; just as the letters may represent the group of the arithmetical symbols, even so they now represent the group of all line segments, and the group of all angles. Thus the algebraic expression, be it a monomial or a polynomial, is cleared of any misconception, and the elements which the science of algebra treats become clearly defined.

Thus far we have considered only the interpretation of the meaning of algebraic symbols or expressions for a correct interpretation of these expressions is absolutely necessary for the study of such relationships between variable quantities as are expressible in algebraic language, but the study of these quantitative relationships themselves is the real concern of algebra. But here again the true import of algebraic language is far from receiving the emphasis which the National Committee recommends. It is not perhaps superfluous to reiterate that the chief aim of mathematics is the study of relationships. In fact is there any field of investigation where the efforts of the investigator are not to find in some way a relation between the factors or variables which enter a phenomenon or a given situation? The use of the function concept in mathematics not only serves as the unifying principle which combines the various branches

of mathematics into one science, but makes the mathematical processes and methods accessible for the understanding of the nature of problems of other fields of study, and the types of relationships which are involved in them. Our consideration of understanding the language of algebra limits our attention to only those relationships which are expressible algebraically, though other mathematical methods, such as graphical and statistical, may be used to advantage in the study of all relationships for which there is no mathematical rule as yet found. But even in the case of the former relationships, their number is legion, and the great majority of them obey the simple laws treated in the secondary school mathematics, *i.e.,* the straight line law, the parabolic law, the hyperbolic law, and the exponential law.

The linear function $ax + b$ gives the straight line laws $y = kx$, and $y = k_1x + k_2$ in terms of which many physical and chemical relationships are expressible. As examples of $y = kx$ we have: $v = n\lambda$, for n constant; $v = at$, for a constant and t variable, or t constant and a variable; $v_r = (v_a/\lambda_a)\lambda_r$, for v_a/λ_a constant; $v_t = (v_0/273)t$, for v_0 constant; $E = 22.4A$; etc., etc. Examples of the law $y = k_1x + k_2$ are: $v_t = v_0 + (v_0/273)t$, for v_0 constant; $\gamma = (M/M_0) - 1$, for M_0 constant; $v = at + v_0$, for a and v_0 constants; etc.

Similarly we could mention many relationships which obey the parabolic, the hyperbolic, and the exponential laws of algebra. But the consideration of the straight line law alone will suffice to bring out the point which bears upon the understanding of the language of algebra. We have already called attention to the fact that the literal numbers should be interpreted as general numbers, variable in character, and that when they enter into a certain relationship which is signified by the equality between certain algebraic expressions, they determine the nature of that law. To illustrate, take three variables x, y, and z and let $y = xz$. Now if x remains constant while y and z vary we have the straight line law $y = k_1z$, and if z remains constant while x and y vary we have again a straight line law $y = k_2x$, but if y is constant and x and z vary we have the hyperbolic law $xz = k_3$. Furthermore, for any one of these three laws the constant k enters as a parameter, that is the particular value that k may have does not change the form of the law (curve), for if it is a straight line it still remains a straight

line, but it does determine the particular straight line which is to represent the law, or the particular ratio which is to be the value of the quotient of the two variables.

Now if the ability to use the language of algebra in the expression of simple quantitative relations means any thing it means that the character of the relationships which are studied in mathematics should be rightly understood. Take for illustration the straight line law $y = kx$. The pupil meets this law first perhaps in the form $d = rt$, and for particular values of r he sees certain relationships such as $d = 30t$, or $d = 10t$, which he is asked to study by evaluating x and y and graphing the resulting data. Suppose he next meets the equation $d = 4t$, does he understand that he is still working with the same law? If he does understand the law then $d = 4t$ will mean to him perhaps the distance that a man walks in t hours if he walks at the rate of 4 miles an hour; $d = 30t$ will apply to a man driving an automobile; $d = 50t$, to a man riding on a train; $d = 100t$, to a man riding in an aeroplane; etc. Since usually the pupil is asked to graph such equations separately, would it not be better to have him graph them using the same lines of reference and the same units? Such a procedure would enable him to see the relation between all these equations, and that they are all different manifestations of the same general law.

Now as far as the mathematical expression of the law is concerned the letters in the formula, being general numbers, may be assigned negative as well as positive values, especially if the field of operations has been extended to include all positive and negative numbers. Many textbooks in algebra are introducing the directed numbers early in the course, which in the opinion of the writer is a desirable feature, and may well be utilized to demonstrate the significance of the constant of the straight line law. In that case, for any positive constant the first and third quadrant are covered with lines, every one of which represents a particular application or manifestation of the straight line law $y = kx$, some of which are real in the sense that they have applications in real life experiences, while others do not have such interpretations. Then assigning the constant a negative value, let it be illustrated for example by a negative acceleration, and the rest of the family of straight lines covering the second and fourth quadrants make their appearance.

Any argument in favor of the postponement of such an interpretation of the straight line law (or the other algebraic laws) until the student takes up Analytic Geometry is neither pedagogically nor logically sound. In the first place, many high school graduates will never go to college, and many that do will not continue their study of mathematics. In the second place, as recommended by the National Committee the understanding of the relationships which are expressible in the language of algebra should be one of the primary aims of the mathematical instruction in the secondary schools, and for a complete understanding of these laws the graphical interpretation sketched above is essential. In the third place, the teaching of graphs in the secondary schools is advocated by all progressive educators, and is undertaken in a more or less perfunctory manner in all courses in algebra and general or correlated mathematics. Hence the suggested procedure offers no new difficulties, but is a systematic utilization of the process toward a very desirable aim, namely, the understanding of the algebraic symbols or expressions which enter into an algebraic relationship or law, and thus the understanding of the language of algebra.

What we have said concerning the straight line law $y = kx$ applies equally well to the interpretation of the law $y = k_1x + k_2$ and the other algebraic laws mentioned above. That they are general laws must be impressed upon the mind of the student, and that they have many practical applications must equally be made clear to him. This will not come automatically by having him solve a few special problems borrowed from physics or chemistry. The ability to interpret the various laws as expressed by the algebraic formula is rather the result of a systematic classification and recognition of the special cases or applications, and the understanding of the nature of the constants or parameters and the variables in terms of which the relationships are stated.

The method of induction is serviceable in mathematics as elsewhere in the establishment of the general law, but induction is certainly not complete when it stops with the consideration of only the special cases without drawing inferences to establish the general law. The emphasis which special cases receive in the textbooks of elementary mathematics is well illustrated by the treatment of the linear equation in one unknown. Page

after page is devoted to the manipulation of this special form of linear equation. The unknown is brought in in all sorts of disguises. All kinds of problems are formulated around the idea of hunting the unknown. Variation is lost sight of, and function concept, which is praised in the preface, is almost an unknown term, and no wonder since it is the unknown which is the hero and receives the lion's share. There can be no functionality where there is no variation.

To devote a major portion of the course in algebra to the manipulation of equations with arithmetical coefficients shifts the emphasis from the study of the general laws to the study of specific cases. Special cases are important and should be freely used, but their study should be a means toward an end, the end being the understanding of the general laws, of which they are but specific applications. Manipulation of special cases or of equations with arithmetical coefficients is but a study of equality and the operations needed to find the value of the unknown quantity.

The study of special cases as illustrated by the equation in one unknown may be used, as stated above, inductively for the determination of the general law. In that case a few examples with arithmetical values might be introduced to understand the nature of the law, then the law might be determined and stated in its general form by the use of literal numbers. Now the general problem is stated, the desired quantity is solved for, and further examples are used as illustrations of the applications of the law.

Consider the applications of the relation between two variables, namely, $y = k_1x + k_2$. If an additional fact, such as $y + x = k_3$, is known, then we have a particular case of the law and x and y now have definite values which involve the constants k_1, k_2, and k_3. The application may be studied in its various forms by taking different sets of arithmetical values for k_1, k_2, and k_3, and for each set a new problem is obtained. Ordinarily the pupil will be asked to solve numerous practical problems all of which are of the same type, that is they involve some particular set of values of the constants k. By this procedure the pupil will have to set up the machinery for each problem anew, whereas if the problem is considered in its general form thinking of k_1, k_2, and k_3 as parameters, the machinery is

constructed only once and the solution of further problems reduces to the recognition of the values of the parameters and the substitution of them in the formula.

The ability to recognize general relationships and to apply them to special situations is one which can be developed better in mathematics than elsewhere. It has been observed by the writer that one of the difficulties which the pupil encounters in the solution of exercises in geometry is due to this very fact. Instead of seeking an established theorem, *i.e.*, a geometric relationship whose truth has been demonstrated, which will apply to the particular problem under consideration, and apply it, he will endeavor to go through the whole demonstration once more and try to prove that relationship again for the special case. If it has been proved that "the diagonals of a rectangle are equal," is it not foolish in proving "the diagonals of a square are equal" to reproduce the whole proof? The proof should consist of but one step: "the square is a rectangle, hence its diagonals are equal," quoting the theorem concerning the rectangle.

Now referring back to the problem stated above in its general form we have for its solution,

Data:

$$y + x = k_3,$$
$$y = k_1 x + k_2.$$

Solution:

$$k_1 x + k_2 + x = k_3,$$

hence

$$(1) \qquad x = \frac{k_3 - k_2}{k_1 + 1}, \qquad y = \frac{k_1 k_3 + k_2}{k_1 + 1}.$$

If instead of $y + x = k_3$ we have $y - x = k_3$, the solution is

$$(2) \qquad x = \frac{k_3 - k_2}{k_1 - 1}, \qquad y = \frac{k_1 k_3 - k_2}{k_1 - 1}, \qquad (k_1 \neq 1).$$

It is interesting to note that in the relations (2) k_1 cannot be unity, for referring to our data we would have in that case the two inconsistent equations $y - x = k_3$, and $y - x = k_2$.

As an example of the relations (1) we may take the problem of dividing s into two parts having the ratio m/n. Before the algebraic solution is taken up the class might be given the two geometric methods of dividing a line segment internally into a given ratio, namely the triangle method, and the parallel lines method. Suppose s is taken to be 40 units, and the ration m/n to be 3/5. First, on the side 40 units (=4cm.) a triangle having the other sides 3cm. and 5cm. is constructed, the vertex angle is bisected, and the resulting segments on the base are measured. Next, by the parallel lines method similar results are obtained and are compared with the first results. Finally, the algebraic solution is considered, and the exact values found by this method are compared with those found by the geometric methods and the per cent of error is computed. Incidentally, it is brought to the attention of the class that whereas the algebraic solution gives exact values, the other two being dependent upon mensuration yield but approximate results.

Once the nature of the problem is understood its general form may be considered and solutions found for the two parts in terms of s, m, and n. Thus,

Data:

$$y + x = s,$$
$$y = mx/n.$$

Solution:

$$\frac{y}{s - y} = \frac{m}{n},$$

hence

$$(3) \qquad y = \frac{ms}{m + n}, \qquad x = \frac{ns}{m + n}.$$

Comparing the relations (3) with (1) it is readily seen that (3) is obtained from (1) by replacing k_3 by s, k_1 by m/n, and k_2 by 0.

The external division of s into the ratio m/n is developed in like manner, and we have for this case

$$(4) \qquad y = \frac{ms}{m - n}, \qquad x = \frac{ns}{m - n}, \qquad (m \neq n).$$

Here again m cannot be equal to n, that is a line segment cannot be divided externally into two equal parts, for the bisector of the exterior angle of an isosceles triangle is parallel to the third side and hence there is no point of division.

Comparing the relations (4) with (2), it is seen that (4) may be derived from (2) in the same way that (3) was derived from (1).

The study of relationships in elementary mathematics gives the course a richness that is apt to stimulate the pupil. Of the many relationships expressible in algebraic language numerous examples could be brought in to illustrate the three important algebraic laws, the straight line law, the parabolic, and the hyperbolic. True enough the modern tendency is to furnish many practical problems as illustrations of these laws, but in general they are considered as mere special cases and hardly ever is the general law determined as a consequence of the problems. An example of the straight line law will suffice to make this point clear. We will first present three separate solutions which are all based on the interpretation of this law, and afterwards will consider the general case where the solution will apply to all sorts of allied problems.

Problem: In an alloy of silver and copper weighing 90 oz., there are 6 oz. of copper. Find how much silver must be added so that 10 oz. of the new alloy shall contain 2/5 oz. of copper.

The law is clearly the straight line law $y = kx$, or $y/x = k$.

Solution (1), considering copper and alloy as variables.

Notation: $c =$ amount of copper,
$a =$ amount of alloy,
and $x =$ amount of silver to be added.

Law: for the original alloy, $c/a = k_1$,
for the new alloy, $c/(a + x) = k_2$.

Data: $c = 6$, $a = 90$, and $k_2 = 2/5/10$.

Solution:

$$6/(90 + x) = 2/5/10,$$

hence

$$x = 60.$$

Solution (2), considering copper and silver as variables.

Notation: $c =$ amount of copper,
$s =$ amount of silver,
and $x =$ amount of silver to be added.

Law: for the original alloy, $c/s = k_3$,
for the new alloy, $c/(s + x) = k_4$.

Data:

$c = 6$, $s = 90 - 6 = 84$, and $k_4 = 2/5/(10 - 2/5) = 1/24$.

Solution:

$$6/(84 + x) = 1/24.$$

hence

$$x = 60.$$

Solution (3), considering silver and alloy as variables.

Notation: s = amount of silver,
a = amount of alloy,
and x = amount of silver to be added.

Law: for the original alloy, $s/a = k_5$,
for the new alloy $(s + x)/(a + x) = k_6$.

Data:

$s = 90 - 6 = 84$, $a = 90$, and $k_6 = (10 - 2/5)/10 = 24/25$.

Solution:

$$(84 + x)/(90 + x) = 24/25,$$

hence

$$x = 60.$$

Now consider the general problem, where the arithmetical numbers are replaced by general, literal, numbers.

General Solution: Applicable to all alloy or mixture problems.

Notation: a = number of parts of original alloy or mixture,

b = number of parts of selected substance in the original alloy,
c = number of parts of new alloy or mixture,
d = number of parts of selected substance in the new alloy,
x = number of parts of the other substance to be added.

Law: $b/a = k_1$, for the original alloy or mixture,
$d/c = k_2$, for the new alloy.

Solution:

$$b/(a + x) = k_2 = d/c,$$

hence

(5) $$x = (bc - ad)/d.$$

It is readily verified that the first solution is a special case of (5), where 60 parts of silver (the other substance, since copper is the selected substance) are to be added. However if we select silver then the solution (5) gives $x = -5/2$, that is 5/2 parts of copper are to be removed. This last value shows that all the alloy of mixture problems containing two substances may be solved by (5), whether the mixture is to be concentrated, or diluted. Furthermore, since when one substance is selected we get a positive answer in the solution, and when the other substance is selected we get a negative answer, the conclusion is that we may obtain the same result, whether of concentration or dilution, by the adding of more parts of one substance or the removal of certain parts of the other substance.

"The territory of arithmetic ends where the two ideas of variables and of algebraic form commence their sway." Variables and algebraic expressions, the essence of algebraic language, make the generalization of specific relationships possible. The operations on algebraic expressions, and the solution of problems can be utilized to bring into prominence the important concepts of variation, relationship, generalization, and application of the law. A course in algebra that does that aids in training the pupil to form the habit of functional thinking, and of using the language of algebra in the expression and interpretation of quantitative relationships. For as Professor Whitehead puts it, "Algebra is the intellectual instrument which has been created for rendering clear the quantitative aspects of the world."

AN ACHIEVEMENT TEST IN SOLID GEOMETRY

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In the work of teaching secondary school mathematics in a large school where there are as many as twelve different divisions of the same subject, it would be very interesting and indeed very enlightening to see the different grades of work being done. Different teachers have their own pet ways of doing things, of presenting new matter, of conducting recitations, of drilling on old matter, of developing mathematical power in their pupils, etc. And yet they are all striving for the same results. The fact that one teacher's pupils consistently attain better results naturally should put a premium on that teacher's methods, and the work of the department would be improved if some of the other teachers would take a leaf out of the successful teacher's book. Students will often remark "So and So is a good teacher; I get a lot out of his class; he makes things clear; he has good discipline; he certainly gets the stuff over, etc." An inspector visits the class, notes the attitude of the pupils, the personality and skill of the teacher, and oftentimes is familiar enough with the subject-matter of the recitation to see if the pupils are catching and giving back the right things, and then grades the teacher as an A1 man, for example. But does the opinion of the boys themselves or the visitor answer the question whether or not the teacher is successful in giving his subject to the pupils? Don't we need something more objective, more tangible, more exact on which to pin our faith? In general the supervisors are hitting it right, also the students, but we think we can do better.

The results of a teacher's work can be discovered quite accurately by a series of comprehensive tests which have been carefully standardized. The College Entrance Examination Board is giving each year cleverly thought-out tests, and the results shown by a teacher's class year in and year out is a pretty good index of the teacher's work. But the examination must of necessity be short, usually confined to six questions taking in the whole

field of Plane or Solid Geometry for example. The element of chance is quite a big factor in the results obtained on these examinations, although it can truthfully be said that if a student is well prepared in a particular subject, he will rarely fail to receive at least a passing grade in the test given by the Board. Also it is quite possible that some comparatively poor teachers may be fairly good coaches for an examination; and it is a well-known fact that skillful tutors will get the proverbial "stick of wood" by the College examination, and then the "stick" will fail to stand up after he gets into College. There is a growing need for more comprehensive and more scientific testing, and where standard achievement tests have been put into use, the results obtained have proved their worth.

Let us now consider in detail what we have in mind for an achievement test in Solid Geometry. We state right here that the test is not the purpose of the teaching; it is merely to find out if the teaching has been done; if any particular weaknesses have been shown along the line; if the students as a whole are capable of grasping the subject at the age when they are studying it, etc. The value of the test must not be overemphasized, and comparatively little time should be given to it. We would suggest two one-period tests during a course in Solid Geometry; one dealing with the relation of lines and planes in space, prisms, and pyramids, and the other dealing with the cylinder, cone, and sphere. Solid Geometry usually being a half-year course, we would have the first test after the subject has been studied about three months. During that time Book 6 and Book 7 in the standard textbooks will have been completed. The other test will come at the end of the course, after about five or five and one half months' work. If we start Solid Geometry in September, we would have the first test at Christmas and the second about the first of March.

When a student has finished Book 6 and Book 7, we should like to test him to see what progress has been made in the development of the following abilities; a knowledge of space perception with the relation of lines and planes in space, a clear visual memory of the main characteristics of prisms, parallelopipeds, cubes, and pyramids, and the mensuration formulas for these more important solids, with the ability to apply them readily. We don't think very much stress should be laid on the formal

proof of the Solid Geometry facts; the student has probably mastered formal demonstration, and had quite enough of it from his work in Plane Geometry. Instead of asking the pupil to give the long rigorous proofs required for four theorems in the Solid Geometry of Books 6 and 7, to be done in a period of forty minutes, we would ask him forty questions which call for a one or two word answer. It is quite evident that we could gain very appreciably in the scope of our questions, in fact forty questions could take in pretty nearly all the important facts necessary for the pupil to retain.

So then, we will have a test divided into three parts and testing three distinct abilities. Part I will be composed of twenty questions calling for a single word answer, and will endeavor to discover if the student has a definite and correct idea of lines and planes in space, perpendicular, parallel, and intersecting planes, etc. Twenty minutes should be plenty of time for this part of the test, one minute per question, but the student will use the time in toto, and therefore may be able to answer some in less than a minute, while he may need more than a minute for others. Part II will try to find out if the student has a visual, working memory of the important solids. If so he will be able to give the one or two word answers at the rate of one per minute. Definition knowledge without formal defining is the ability tested in Part II. Part III calls for some of the important mensuration formulas with direct application to simple problems, where the fact and its application is the important part and the number manipulation comparatively simple. This ability, we think, is the main justification for the teaching of Solid Geometry.

In each part there is an attempt to grade the questions from the simple to the more difficult, but not to hold to any logical sequence. In fact we wish to make the student jump around, to see if he can call up the situation demanded in each question with accuracy and dispatch, and then go on to another and do the same thing in an entirely different situation. In all probability there will be enough matter to test the most brilliant in the class; on the other hand there are enough easy questions so that the poorest in the class should make some kind of showing. The answers are all short and exact, calling for no particular judgment on the part of the reader of the paper. The scoring

of an individual paper should be a matter of a few minutes, the results can easily be gotten back to the teacher and pupil, and whatever weaknesses they show can subsequently be remedied. There are no norms for this particular test, but we propose to give it to enough pupils so that we can get reliable norms.

The test itself follows, including the directions for each teacher who is to give it to his division, and also the directions to be read by each teacher to the class he is testing. Each student is to be given an exact replica of the test as it appears on the succeeding pages.

Directions to Teachers

As soon as the final bell rings, have the desks cleared of everything. Put one test on each desk, saying to the class, "Please don't touch these papers until I give you definite directions." As soon as you have distributed the tests, stand at the desk and read the directions. The test should begin within three minutes after the final bell. Take the attendance after the test has started. Please watch carefully for any communicating, and if you see any, immediately take the paper. Write "Communicated" at the top of the first page of paper and send it in with the rest. At the end of the test, collect the papers yourself, making sure that you get one from each person. Make sure that all tests given out come back. As soon as you have collected all the tests put them immediately in the envelope and close the envelope. Please don't look at or discuss any test in any way after it is once finished.

Directions to be Read by Teacher to Class

Attention! Follow directions carefully. Use pencil. Look at page 1. Write your name where it says name, last name, and first name in the proper place. Write class symbol in space indicated (teacher gives symbol as M32Mc). Write the date, Jan. 14, 1927. Pencils up! You are now going to be tested on the Solid Geometry you have learned this year. Do the very best you can because your work will be compared with the work of other students in this school, and perhaps in other schools. You will probably not have time to finish all the questions on the following pages. Do as many as you can, but don't hurry. Ask no questions. Don't communicate. You are to turn the sheet and begin when I say "Go." One minute before the end of the period I will say "Stop! Pencils up!" All pencils will come up immediately and papers will be collected. You will use no scratch paper. Any figuring you may wish to do may be done in the margin of the test. Keep your eyes on your own paper. READY! GO!

PAGE 1

English High School, Boston. Date____________________

Name__________________________ Class____________________
Last name First name

PAGE 2

Part 1

Directions.—In answering each question in this part of the test, underline the word in the parenthesis which you think answers the question correctly.

1. Two planes intersect in a (curved line, point, straight line).
2. How many planes may be passed through a straight line? (two, one, any number)
3. How many planes may be passed perpendicular to a straight line? (two, one, any number)
4. How many planes may be passed through a point in a straight line perpendicular to the line? (two, one, any number)
5. How many planes may be passed through a point outside a straight line perpendicular to the line? (two, one, any number)
6. How many planes may be passed through one of two parallel lines parallel to the other? (two, one, any number)
7. How many lines may be drawn perpendicular to a plane from a point outside? (two, one, any number)
8. Are two lines which never meet always parallel? (yes, no)
9. Are two planes which never meet always parallel? (yes, no)
10. How many lines may be drawn perpendicular to a plane? (two, one, any number)
11. If two lines are not in the same plane, how many planes may be passed through one of them parallel to the other? (two, one, any number)
12. How many lines may be drawn through a point in a plane perpendicular to the plane? (two, one, any number)
13. How many lines may be drawn perpendicular to a line at a given point in the line? (two, one, any number)
14. Two planes perpendicular to the same straight line are (perpendicular, parallel, intersect)
15. If two parallel planes are cut by a third plane, the intersections are (perpendicular, parallel, intersect)
16. How many planes may be passed perpendicular to the given plane through a given line perpendicular to the given plane? (two, one, any number)
17. Through a line not perpendicular to a given plane, how many planes may be passed perpendicular to the given plane? (two, one, any number)
18. How many equal oblique lines may be drawn to a given plane through any point in a perpendicular to the given plane? (two, one, any number)

19. Is an oblique line from a point to a plane shorter than a perpendicular from the same point to the same plane? (yes, no)

20. How many lines can be drawn perpendicular to each of two given lines which are not in the same plane? (two, one, any number)

Part 2

Directions.—Write in the parenthesis after each question what you think is the correct answer.

1. How many lateral faces has a parallelopiped? ( )
2. How many edges has a cube? ( )
3. How many vertices has a triangular pyramid? ( )
4. Is a right section of an oblique prism perpendicular or oblique to the lateral edges? ( )
5. How many rectangular faces has a right parallelopiped? ( )
6. How many vertices has a pentagonal prism? ( )
7. How many faces of a hexagonal prism are parallelograms? ( )
8. The plane passed through two diagonally opposite edges of a parallelopiped divides it into—(Finish)— ( )
9. A section of a pyramid made by a plane parallel to the base is ——————— to the base. (Put the missing word in the parenthesis.) ( )
10. An oblique prism is equivalent to a right prism whose base is a—(Finish)— ( )

Part 3

Directions.—Write in the parenthesis after each question what you think is the correct answer.

1. What is the formula for the volume of a pyramid? ( )
2. The area of the base of a square pyramid is 40 sq. ft. and its volume is 120 cu. ft. How high is it? ( )
3. A prism and pyramid have equal bases and equal volumes. How does the altitude of the prism compare with the altitude of the pyramid? ( )
4. The area of a section of a pyramid parallel to the base is one fourth the area of the base. How much of the altitude is cut off by the section? ( )
5. The area of one face of a cube is 12 sq. in. What is the total area of the cube? ( )
6. A regular prism has pentagons for its bases. Its altitude is 8 in. and its volume is 96 cu. in. What is the area of each pentagon? ( )
7. What is the volume of a cube whose edge is 4? ( )
8. The bases of two rectangular parallelopipeds are equal. The altitude of the first is 4 and of the second 16. What is the ratio of the volume of the first to that of the second? ( )
9. The perimeter of a right section of an oblique prism is 20 in. The lateral edge is 8 in. What is the lateral area? ( )
10. What is the altitude of a pyramid whose volume is 200, and whose base is a rectangle 5 by 12? ( )

The test was given January 14, 1927 to twelve divisions in Solid Geometry in English High School, Boston, Mass. The total number of boys who took the test was 376. The time allowed was one teaching period of forty-five minutes, and if we take into account the three minutes necessary to get ready, read instructions, etc., at the beginning, and the one or two minutes at the end, the student actually had forty minutes in which to perform. The test was "sprung" on all divisions; they had no

TABLE I

RAW SCORES

Number of Rights	Number of Pupils
40	0
39	1
38	3
37	6
36	8
35	8
34	15
33	6
32	11
31	18
30	14
29	20
28	22
27	17
26	22
25	22
24	20
23	21
22	19
21	20
20	22
19	21
18	13
17	11
16	12
15	6
14	8
13	3
12	3
11	1
10	1
9	1
8	1
7	0
	376 Totals

warning, no special preparation. It was the consensus of opinion of the six teachers who gave the test that thirty minutes would have been ample time as a number of the pupils were finished about ten minutes before the time was up.

The results of the test can best be summed up in the following tables, each one of which we will try to explain in as few words as possible. Table I shows the raw scores obtained by the pupils. There were no perfect scores, that is, no student answered the forty questions correctly. One pupil answered thirty-nine, three, thirty-eight, etc. The poorest pupil answered eight correctly. The distribution can readily be seen from the table.

TABLE II

Rights.	Div. 1	Div. 2	Div. 3	Div. 4	Div. 5	Div. 6	Div. 7	Div. 8	Div. 9	Div. 10	Div. 11	Div. 12	Totals.
40													
39	1												1
38							1		2				3
37				4				1	1				6
36				3		1		1	1	1		1	8
35	2	1		1	2							2	8
34	4	1		2					5	2		1	15
33	1		2	1						1		1	6
32	2		2	1			2		1		1	2	11
31	1	3	3	3	3	1		2	1			1	18
30		3		1		1	2	1	2	1		3	14
29	2	1	3	1	2		1	1	4	1	1	3	20
28	3		4	3	1	1		4	2	2	1	1	22
27			2	2	3	2		3	2	3			17
26	1	2	1	1	3	4	1	2	2	2	3		22
25	2	1	1		1	3	4	2	2	4		2	22
24		4	3	2	1		3			2	3	2	20
23		4	3	1	2	3	1	3		1	1	2	21
22	2		1	2	3	1		3	1	3	2	1	19
21	1	1	3		5	1	5			2		2	20
20	4	1	1	1	1	7	1		1	1	1	3	22
19	2	2	3	1	3	2		1	2	3	2		21
18	2		4			2	2			3			13
17	1	1	2	1		1	2	1		1	1		11
16		2			1	1			2	1	3	2	12
15		3				1	1			1			6
14					1	1	1		1		4		8
13							1		1		1		3
12							1			1	1		3
11							1						1
10						1							1
9					1								1
8								1					1
7													
Totals	31	30	38	31	33	30	26	34	33	36	25	29	376

A = 11%. B = 24%. C = 46%. D = 17%. E = 2%.

Table II is a distribution of the raw scores as they were obtained by the members of each one of the 12 different divisions. The divisions are numbered across the top, the number of rights obtained down the left-hand column, and the number of pupils obtaining a certain number of rights in the column headed by their division number. For example in the column headed Div. I, one pupil had 39 rights, 2 pupils 35, 4 pupils 34, etc. The total numbers of pupils in the division are written along the bottom of the table. The column at the right is the total raw score column with a percentage distribution of grades A, B, C, D, E, those used in the Boston Schools. Those pupils obtaining from 34 to 39 inclusive correct answers, or the upper 11 percent might be given the mark "A," the next group "B," etc. The heavy rules across the page separate the "A's" from the "B's," the "B's" from the "C's," etc.

Table III represents the number of "A's," "B's," "C's," "D's," and "E's" obtained by the students of the six different teachers. For example, teacher C had 4 pupils in the "A" group, 15 in the "B," 26 in the "C," 10 in the "D" and 1 in the "E." His total number of pupils was 56 and he had two divisions.

TABLE III

Teacher	Number of Marks					Totals	Divisions
	A	B	C	D	E		
Mr. A.......	19	20	17	8	0	64	2
Mr. B.......	11	20	22	7	0	60	2
Mr. C.......	4	15	26	10	1	56	2
Mr. D.......	2	20	34	14	1	71	2
Mr. E.......	2	11	46	26	4	89	3
Mr. F.......	3	5	18	9	1	36	1
Totals......	41	91	163	74	7	376	12

Table IV represents the percentages of A's, B's, C's, D's, and E's, obtained by the student of the six different teachers.

Table V shows the distribution of raw scores into a smaller number of groups for the purpose of obtaining the exact median, the upper quartile, and the lower quartile of the entire 376 students. The column at the left represents the group division, the width of the group being 2.5, and the other column the number of scores in each group.

TABLE IV

Teacher	Percentages of Marks				
	A	B	C	D	E
Mr. A	30	31	27	13	0
Mr. B	18	33	37	12	0
Mr. C	7	27	46	18	2
Mr. D	3	28	48	20	1
Mr. E	2	12	52	29	4
Mr. F	8	14	50	25	3
Whole group	11	24	46	17	2

TABLE V

Groups	No. of Scores	
37.5–40.0	4	
35.0–37.4	22	Median score 25.12
32.5–34.9	21	
30.0–32.4	43	
27.5–29.9	42	Upper quartile 29.66
25.0–27.4	61	
22.5–24.9	41	Lower quartile 20.53
20.0–22.4	61	
17.5–19.9	34	
15.0–17.4	29	
12.5–14.9	11	
10.0–12.4	5	
7.5– 9.9	2	
	376	

Table VI shows the median and quartile variation of each one of the 12 divisions.

Div.	Median Score	Upper Quartile Score	Lower Quartile Score
1	28	33	20
2	24	30	19
3	24	29	19
4	30	35	24
5	23	28	21
6	22	26	19
7	27	29	23
8	23	26	18
9	29	34	22
10	24	27	19
11	28	32	21
12	20	26	16
Group	25.12	29.66	20.53

Median range, 30–20, 10 points; upper quartile, 35–26, 9 points; lower quartile, 24–16, 8 points.

The test was corrected by two divisions of Solid Geometry students, about thirty in a division, working one period each. The results of his own groups were given to each teacher with the percentage attainment of the entire group, and he could use the marks obtained as part of his regular bi-monthly ratings at his discretion. He could see just what each of his students did, in what ability he was weak, and of course could use the corrective measures necessary.

From a diagnostic point of view, the test showed that the students as a whole were fairly strong on the general notion of lines and planes in space and the number of parallels and perpendiculars possible under certain conditions. They were rather weak on visual memory of the more common solids; for example, they undoubtedly knew what a cube was, and yet couldn't count its edges with a high percentage of success. They were weak on the simple numerical application of the mensuration formulas for prisms and pyramids; in a large number of cases they knew the right formula but failed to obtain the right answer. Students can never get too much drill on numerical accuracy.

It was quite evident from the reactions of the students that they were very much interested in the new type of test. It was a complete and quite agreeable change from the ordinary one of two theorems to be proved and two long numerical exercises to be worked out. There seems to be a peculiar fascination for students to answer a large number of questions and then be very anxious to know how many they had correct. A large number of the students asked spontaneously, "When are we going to have another?" They were very eager to compare their work with the work of other divisions, and each one seemed quite anxious to know in just what section of the total number his achievement placed him. The teachers also were very much interested to know where each one of their groups placed in the large group. We are quite sure that they would be willing to have another, and also that they would probably teach just a little harder to raise the median achievement of their divisions.

As a result of our study and our experiment in testing a large number of students with a different kind of test, the natural question is how shall we make more effective our present work in the classroom, and how shall we make better use of tests in the future to improve our work. We might suggest that a group of

teachers in a mathematics department get together, decide on the minimum essentials in a certain subject at a certain period of the work, and by each one contributing a number of questions, frame a test composed of a large number of questions, and give it to all the students studying the subject at the same time. Then the marks given would have some significance; they would not be unreliable as teachers' marks notoriously are. Each teacher would be more alert to the difficulties of the various parts of the subject-matter, would soon learn where his presentation has been weak, and sometimes find out what material is really too difficult for the calibre of his class. In short he would do quite a little healthy analyzing of his own work, and receive a lot of real constructive criticism from the results of these more scientific tests. Teachers after all are only human and now and then an extra incentive to good work is quite welcome. The students also would be stimulated to do better work, and once we get them eager, the rest is easy.

It seemed to be the consensus of opinion of all the teachers concerned that such a test as the one given was a very happy relief from the type ordinarily used, and that it would serve to give a little interest and impetus to the teaching of perhaps the most uninteresting and difficult branch of secondary school mathematics, Solid Geometry.

ISOTOMIC POINTS OF THE TRIANGLE

By PROFESSOR RICHARD MORRIS,
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In *A Sequel to Euclid*, fourth edition, page 168, Dr. John Casey defines Isotomic lines and points. See Fig. 1. Take

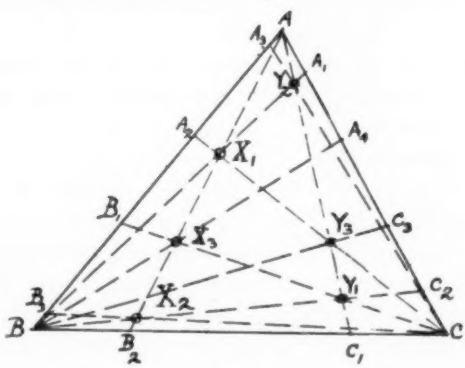

Fig. 1.

points B_2 and C_1 on the segment BC such that BB_2 equals CC_1. Such lines are called by Dr. Casey, Isotomic Conjugates with respect to angle A. Take points C_2 and A_1 on the segment CA such that CC_2 equals AA_1, not necessarily equal to BB_2, though they may be. The intersection of lines AB_2 and BA_1 locate point X_1 and the intersection of lines BC_2 and AC_1 locate point Y_1. Then lines CX_1 and CY_1 locate points A_2 and B_1 on segment AB such that AA_2 equals BB_1. Using Ceva's theorem, we get

$$\frac{AA_2}{A_2B}\cdot\frac{BB_2}{B_2C}\cdot\frac{CA_1}{A_1A} = 1 \qquad \text{and} \qquad \frac{AB_1}{B_1B}\cdot\frac{BC_1}{C_1C}\cdot\frac{CC_2}{C_2A} = 1.$$

Multiplying these two expressions together and cancelling the quantities denoting equal segments, we get

$$\frac{AA_2}{A_2B}\cdot\frac{AB_1}{B_1B} = 1$$

or

$$\frac{AB_1}{BB_1} = \frac{A_2B}{AA_2}$$

or

$$\frac{AB}{BB_1} = \frac{AB}{AA_2} \qquad \text{(By addition).}$$

Whence

$$BB_1 = AA_2.$$

Points such as X_1 and Y_1 that have the relation set forth in the above demonstration are called Isotomic conjugate points of the triangle. Depending upon the combination of conjugate rays employed, a variety of conjugate points may be obtained, such as X_2, Y_2 and X_3, Y_3, etc., the proof of which is similar to the above.

It is evident that the Centroid, G, of the triangle ABC is its own Isotomic conjugate, and it is the only point of the triangle which enjoys this relation. But every point other than G does have a conjugate.

The equal distances may be taken between the vertices, or some may be taken beyond the vertices. However, of the three sets of equal distances, either one pair or all three pairs must be taken between the vertices. If any distances are taken beyond the vertices, there must be exactly two pairs. The position of the equal segments shows that the conjugate points must both lie inside the triangle or both outside; there cannot be one inside and one outside. If the conjugate points lie outside the triangle, they lie on opposite sides of a side of the triangle. The isotomic lines lying inside the triangle are extended in opposite directions from the vertex to locate the conjugate points on opposite sides of the side of the triangle. There may be a pair of conjugate points in any one of four positions, viz., (*a*) both inside, (*b*) one on either side of BC in angle A, (*c*) one on either side of CA in angle B, (*d*) one on either side of AB in angle C. Illustrations: The conjugates of the orthocenter, the circumcenter, and the incenter are inside the triangle, while the conjugates of the three excenters of the triangle illustrate the other three cases.

The isotomic conjugate of a point on the side of the triangle is the opposite vertex. The conjugate of a vertex of a triangle may be any point on the opposite side. The conjugate of a side of the triangle is a side of the triangle.

Problem.—The feet of the altitudes of a triangle determine distances from the vertices on the corresponding sides of the triangle. If AD' (see Fig. 2) is the conjugate of AD, and if

AK and BM are made equal to AF, prolonging AD' and DA respectively, we locate a pair of conjugates H_A and H_A' analogous to H and its conjugate H' inside the triangle. If however, AD and $D'A$ are prolonged, another pair of points is located nearer the triangle, such as H_1 and H_2. But the distance AF might just as well have been laid off on side AC instead of on AB, so that we would have two additional pairs of isotomic conjugates in the same relative position.

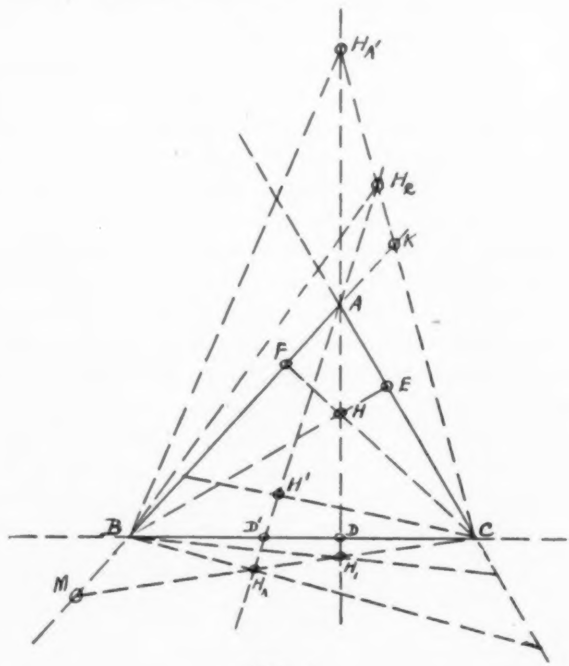

FIG. 2.

If the distance AE is used in the same manner that AF was used, there is obtained four more pairs of conjugate points in the same region as the former four pairs.

We may do the same thing with the distances determined by the altitudes BE and CF, making in all twenty-four pairs of conjugate points outside the triangle, analogous to the pair H and H' inside the triangle.

This same process may be applied to any point and obtain a variety of pairs of conjugate points. But, if the centroid, G,

being its own conjugate, is treated in the manner described above, the number of pairs reduces to twelve, each median containing four pairs, besides the centroid itself.

If, in an isosceles triangle, AB and AC being the equal sides, the conjugate lines from B and C cut off four equal segments, the third pair of conjugate lines coincide with the altitude from vertex A and the conjugate points are on this altitude, for the points can be shown to lie on the perpendicular bisector of side BC. But if the pairs of segments are not equal, the conjugate points lie on opposite sides of the altitude.

In an equilateral triangle the three pairs of conjugate points determined as for the isosceles triangle lie on the altitudes (medians). If the alternate points are joined, two equilateral triangles are formed but these are not congruent. If the points are joined consecutively, the hexagon so formed has its sides equal, but the figure is not regular.

Problem.—(See number 2, page 169, *A Sequel to Euclid,* Casey.) If a line p cuts the sides AB, BC and CA of a triangle ABC, in points C', A' and B' respectively, then the isotomic conjugates of the lines to these points cut the same sides in points C'', A'' and B'' which are collinear in line q. See Fig. 3. The proof

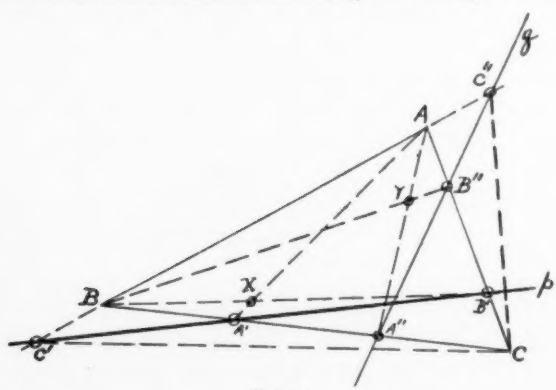

FIG. 3.

is as follows: The rays AA'' and BB'' are the conjugates of AA' and BB' respectively. Let C'' be the point where the straight line $A''B''$ cuts side BA. We have to show that AC'' equals BC'. Using the theorem of Menelaus for line p we have

$$\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = -1,$$

and using the theorem for line q, we have

$$\frac{BC''}{C''A} \cdot \frac{AB''}{B''C} \cdot \frac{CA''}{A''B} = -1.$$

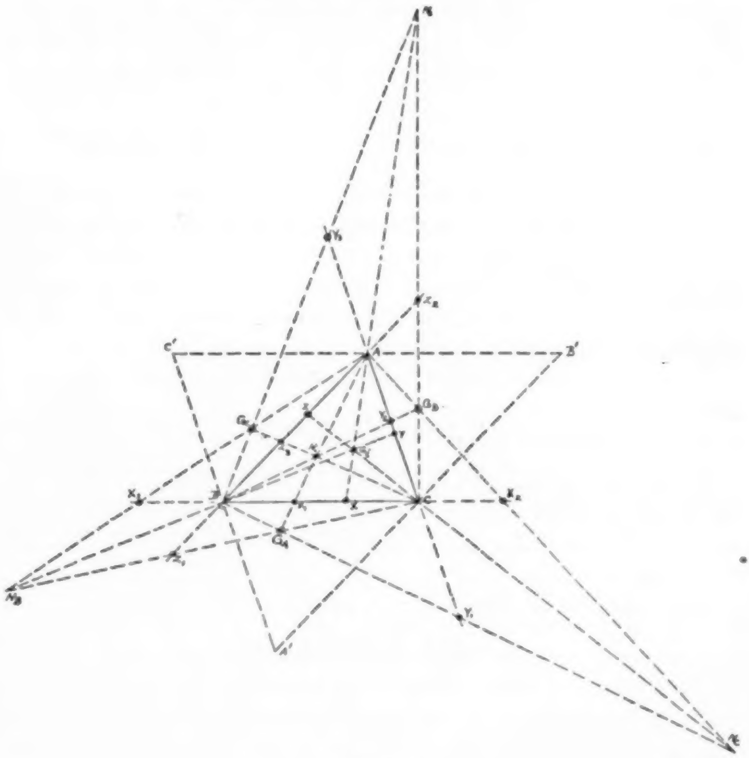

Fig. 4.

Dividing the first by the second and cancelling the equal segments, there is obtained

$$\frac{AC'}{C'B} \cdot \frac{C''A}{BC''} = 1$$

or

$$\frac{AC'}{BC'} = \frac{BC''}{C''A}$$

or

$$\frac{AB}{BC'} = \frac{AB}{C''A} \qquad \text{(by subtraction).}$$

Hence

$$BC' = C''A.$$

If line p passes through a vertex, line q coincides with the opposite side, since the conjugate of a side of a triangle is a side of the triangle.

A very interesting application of isotomic conjugate lines and points is in connection with the points of tangency to the sides of a triangle of the in-circle and the three ex-circles. See Fig. 4.

Let X, Y and Z be the points of tangency of the in-circle on sides BC, CA and AB respectively.

In the identity $2s = a + b + c$, we get by substitution,

$$2s = a + AY + YC + AZ + BZ,$$
$$2s = a + AY + AZ + CX + BX,$$
$$2s = 2a + 2AY.$$

Hence

$$AY = AZ = s - a.$$

Similarly

$$BZ = BX = s - b$$

and

$$CX = CY = s - c.$$

Let X_1, X_2 and X_3 be the points of tangency of the three ex-circles on the side BC, Y_1, Y_2 and Y_3 the points of tangency on the side CA and Z_1, Z_2 and Z_3 the points of tangency on the side AB.

Again, taking $2s = a + b + c$, we get by substitution,

$$2s = BX_1 + CX_1 + b + c,$$
$$2s = b + CY_1 + c + BZ_1,$$
$$2s = AY_1 + AZ_1 = 2AZ_1.$$

Hence

$$AZ_1 = AY_1 = s.$$

Similarly,

$$BX_2 = CX_3 = BZ_2 = CY_3 = s.$$

Evidently,

$$BZ_1 = BX_1 = AZ_2 = AY_2 = s - c.$$

Hence, there are six segments which equal $s - c$. Similarly

$$BX_3 = BZ_3 = CX_2 = CY_2 = AZ = AY = s - a,$$

and

$$AY_3 = AZ_3 = CY_1 = CX_1 = BX = BZ = s - b.$$

Now these distances from the vertices of the triangle along its sides are such that the application of isotomic lines to the

points of tangency give very interesting results. It is possible to find four pairs of isotomic conjugate points.

The Gergonne Points

If the vertices of the triangle are joined to the points of tangency of the in-circle, we may show by Ceva's theorem that these lines are concurrent inside the triangle. Thus

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{s-b}{s-c} \cdot \frac{s-c}{s-a} \cdot \frac{s-a}{s-b} = 1.$$

This point, G_i, known as a Gergonne point was discovered more than one hundred years ago. If the vertices of the triangle are joined to the points of tangency of the ex-circle whose center is opposite vertex A, we obtain the Gergonne point G_A, which is outside the triangle and on the opposite side of BC from A. The points G_B and G_C are found in the same manner.

The Nagel Points

If the vertices are joined to a point of tangency of the three ex-circles, we show by Ceva's theorem that these three lines are concurrent inside the triangle. Thus

$$\frac{BX_1}{X_1C} \cdot \frac{CY_2}{Y_2A} \cdot \frac{AZ_3}{Z_3B} = \frac{s-c}{s-b} \cdot \frac{s-a}{s-c} \cdot \frac{s-b}{s-a} = 1.$$

This point, N_i, is known as a Nagel point and was discovered less than one hundred years ago. If A is joined to X, a point of tangency of the in-circle, and the line is produced through A, and if B and C are joined to Y_3 and Z_2 respectively, these three lines are concurrent in N_A, a Nagel point in angle A but outside the triangle. Similarly, N_B and N_C are located in angles B and C respectively, but outside the triangle. These eight points were studied at length the latter part of the 19th century by E. Vigarié. The present writer would point out that these points form four pairs of isotomic conjugate points.

Since we have shown earlier that AX and AX_1, BY and BY_2, and CZ and CZ_3 are pairs of isotomic conjugate lines, it is evident that G_i and N_i are conjugate points. Again, since AX and AX_1, CZ_2 and CZ_1, and BY_3 and BY_1 are pairs of isotomic conjugate rays, the points G_A and N_A are also conjugate points. Similarly G_B and N_B and G_C and N_C are pairs of conjugate points.

It should be noted that the Gergonne points are obtained by joining the vertices to the three points of tangency of one circle, but that in the case of the Nagel points, the vertices are joined to the points of tangency, one taken from each of three circles of the four.

If two pairs of conjugate rays are used other than those we have already employed, a variety of isotomic conjugate points may be obtained, both inside and outside of the triangle, but of the points so obtained none are notable points, except the centroid, which is its own isotomic conjugate, and the eight Gergonne-Nagel points.

The line Z_1X_1 determines a point on AC whose distance from C is equal to the distance from A on AC of the point determined by the line XZ_2.

Likewise, if any pair of points of tangency is joined, isotomic distances are determined on the sides of the triangle, there being just 24 such pairs of isotomic distances. The lengths of these isotomic distances determined by these lines through two points of tangency may be computed, using the theorem of Menelaus.

It should be mentioned that the Nagel points are the centers of the in- and the three ex-circles of the anticomplementary triangle, i.e. the triangle formed by drawing through the vertices of triangle ABC lines parallel to the opposite sides. The triangle $A'B'C'$ is the anticomplementary triangle of the triangle ABC. This fact has been known for forty years or more, but the proof is not suitable for this article.

In the preparation of this paper, the writer has used *A Sequel to Euclid*, by John Casey, *College Geometry*, by Altshiller-Court, and *A Treatise on the Circle and the Sphere*, by J. L. Coolidge.

TECHNIQUE AND DEVICES CONDUCIVE TO BETTER TEACHING OF GEOMETRY

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The difficulties facing the teacher of plane geometry grow constantly more baffling. Enforced attendance at school until the boy or girl is sixteen, with a group of pupils often unfitted mentally, by environment, or by ambition for a high school education, presents an immediate and ever-looming problem. An insufficient number of trade schools or excessive ambition on the part of generous parents, desirous of seeing their children in the professions rather than in the trades, fills the classes of such a subject as plane geometry. Moreover the subject is generally required for college entrance, and is regarded almost universally as cultural, broadening, and conducive to mental development characterized by clear expression and logical thinking. The position of the teacher of this subject fifteen or twenty years ago when a select few with real ability pursued the subject was not nearly so complex. Now every youth who has managed to pass to the tenth grade and takes up plane geometry all but assumes that a passing understanding of the subject is his birthright. Perhaps it would be conceded that plane geometry is the first great obstacle to the youth's securing a coveted diploma. Or possibly it might be stated thus: it is in his attack upon this subject in which the frailty of his mental make-up is most in evidence, most pitilessly laid bare, if there is such weakness; or, on the other hand, his power of intellect is here first appreciated by others, and, with great satisfaction, by himself. Hence the instructor of plane geometry to-day who would teach the subject in a forceful and effective manner, developing his pupils, convincing them of its influence, its use and keen intellectual enjoyment, must be forever on his toes inventing and contriving devices and stratagems to teach satisfactorily this large group, less clever, less ambitious, less able, than heretofore, taken in the cross-section. What a wonderful people the citizens of the United States would be intellectually several generations hence

if all who attempt a high school education really had the ambition and ability to master it.

It is perhaps axiomatic that it is desirable to select in geometry, first of all, a text in which the theorems are demonstrated at the beginning in full, but gradually with more and more in the way of suggestion and omission that the pupil may use and develop by degrees his fund of information, power of reasoning and originality. Moreover this expedient tends to prevent his memorizing the proof verbatim. The parallel form of formal proof is most satisfactory. This method divides the paper by means of a vertical line through the middle, presenting the demonstration in two columns, the one on the left showing the successive steps or statements of the proof, the one on the right the consecutive reasons or authorities paralleling the steps. From the outset the pupil should be required to use consecutive letters different from those of the text to obviate the tendency to memorize the demonstration letter for letter.

As soon as the pupil has progressed far enough to attempt simple exercises and corollaries, an outline of suggestions for studying them should be presented. The following offers assistance:

I. Copy the theorem or statement.

II. Digest every word of the theorem, recalling or looking up each geometrical term used.

III. Draw or construct the figure carefully and go over it to see that it satisfies the theorem.

IV. Write the hypothesis.

V. Ask yourself what you know about every word in the hypothesis. Recall every authority that might lend meaning to the hypothesis.

VI. Write the conclusion.

VII. Decide what method of proof seems most appropriate:

A. Direct,
B. Superposition,
C. Indirect,
D. Analysis,
E. Combination of two of these.

VIII. Plan the proof mentally, keeping constantly in mind the hypothesis from which you are going and the conclusion, the

goal toward which you are aiming. If the proof suggests being long or complex, jot down on a scratch pad all facts that might have a bearing on the proof. Later assort and arrange such facts in an order that proceeds from the hypothesis to the conclusion.

IX. Write out the proof in full, giving an authority or reason for each step.

The pupil should be required as early as possible, consistent with the work mastered, to construct each figure. He should be warned that the appearance of figures, even of those constructed, is often misleading. He should be shown some optical illusions so as to convince him of the fact. The habit, as soon as the theorem is read and understood, of constructing the figure with reasonable neatness in as general form as the hypothesis requires tends to emphasize precisely that hypothesis and as well the conclusion required. A faulty or particular figure may lead to inaccurate results. The pupil should be taught to study, with pencil and paper constantly at hand, sketching or constructing figures even if his attention is centered upon a mere definition. He should learn to construct perpendiculars in all sorts of positions, to the upper edge of the board, the lower edge, the lateral edge, and to oblique lines. He should learn when he bisects an angle to bisect an obtuse angle and a reflex angle as well as the usual acute angle. In a short time a mere glance at a correctly constructed figure should tell him almost as much as his hypothesis. He should realize that the figure and hypothesis are complementary, the hypothesis, however, being the criterion. He should identify constructions among a group of different sorts and realize at once if one arc or a pair of arcs is incorrect.

He should be drilled in separating the hypothesis from the conclusion in numbers of theorems familiar or not yet attempted as proofs. Often the device of underscoring the hypothesis with chalk of one color whereas the conclusion is underscored with chalk of a different color, using green regularly for the hypothesis and red consistently for the conclusion, these very colors suggestive, assists in emphasizing the two concepts, linked, yet separate, one a natural outcome of the other.

The study of geometry can be made far more interesting, more familiar and often more meaningful by introducing algebraic notation wherever it is possible. The pupil is well acquainted

with algebra and feels himself on familiar ground when dealing with it. Theorems translated to three or four letter formulas, whenever possible, are of easy application. A line segment designated by a single letter, "l," can be squared or multiplied with ease and dexterity. The product of two line segments each represented by a letter should produce for the pupil a very distinct result geometrically, namely, a plane. The square of a line segment should represent an area geometrically, a square. The product of a line segment by an abstract number should represent a line segment. He should know what sort of geometrical result to expect if he divides the product of two line segments by an abstract number or by another line segment. In this way his algebra will take on new and more specific meaning and his geometry will be not wholly foreign. Moreover the use of algebra usually simplifies the procedure to a marked degree. The parts of Euclid's Elements which we have retained for text book use to-day convey a disproportionate notion of his use of algebra, cumbersome as his notations were.

Concerning the proofs of the theorems, more or less complete in the text, the pupil should be questioned as to what in general is accomplished in the various steps, what the skeleton of the proof is, what each of the various steps attains in the way of advance toward the conclusion and how that result is secured finally. He should try to follow through the demonstration mentally, step by step, holding in mind a picture of the figure without naming it with specific letters, stopping after each step to supply the necessary reason. Again, an oral demonstration should often be given before the class, no points of the figure being named with letters, the pointer alone indicating the part of the figure under consideration at any one moment. Often the question should be raised of interchanging various steps of a deductive proof. Can any be interchanged, and if so, how? Moreover there is the matter of connections. Is "and," "but," "hence," "therefore," "also," "moreover," "whereas" or "whence" more appropriate or effective here or there? And why? Such discussions and criticism lead to genuine coherence and sequence, in fact, real reasoning, which carries over to other studies and life habits in thinking.

Often in a proof, classed by the instructor as deductive or synthetic, there is a main current of reasoning, then a sharp

detour, later a gradual joining of the second sequence of argument with the main line of reasoning. The purpose of this detour should be explained, the point at which it is made and the argument by which the main theme is again reached, all such discussion being made in general terms, not with particular letters. This type of demonstration is extremely difficult for the novice, as he is familiar with and prefers only a single thread of reasoning.

As soon as the pupil has progressed far enough to have worked with both direct and superposition proofs, their respective types, ear-marks and purposes should be pointed out. The Latin derivation of the word "superposition" should be discussed, and the fact that this method is used only in a few fundamental proofs where no other type of demonstration is possible. The synthetic or deductive proof known to the pupil as the "direct proof" should be regarded as the usual and most universal type, reasoning directly from hypothesis to conclusion. All of his theorems should be classed under one or the other type until he reaches a combination of the two methods in the theorem concerning the congruence of triangles given three sides of one respectively equal to three sides of another. Later, with the advent of the indirect demonstration, it, as a type, should be studied carefully, as to plan, use, the manner of drawing the suppositional line, and the fact that the figure can not be constructed so as to conform to both the demands of the hypothesis and also those of the supposition, but that the construction made so as to satisfy the hypothesis is always much more satisfactory, if at all possible. Moreover it should be emphasized that the success of the indirect proof depends upon whether or not one has taken into account each and every possible conclusion of a given hypothesis other than that of the theorem under consideration.

In this early work of the pupil when the proofs are so confused and confusing to him, when there seems to be no rhyme or reason to it all, there are a few simple devices that tend to guide him to gradual understanding of what it is all about. The trick of placing a check mark on each of a pair of equal parts of a figure jogs the memory. A second pair of equal parts is branded with a pair of double check marks and a third pair of equal parts, each with triple check marks. Another contrivance, that of checking off the successive parts of the hypothesis just after each is actually

put to use in a demonstration, is available. Otherwise the hypothesis is not exhausted as to use in the demonstration but left forgotten above the attempted proof. The same device of checking off conclusions, as proved, if there are two or more, is most helpful. In the superposition proof the habit of drawing one figure with colored chalk whereas the other is drawn with white, then placing the colored one on the white, coloring over the white lines, step by step as the various new positions are justified by reasoning, is conducive to genuine understanding and tolerance for this type of demonstration, at best tedious. Colored chalk is invaluable in constructing auxiliary lines, added in the course of the proof, in contradistinction to those of the hypothesis, in such a proof as the one concerning angles the sides of which are perpendicular each to each, or the theorem concerning two triangles having two sides of one respectively equal to two sides of the other, but the included angles unequal, or the theorem concerning a series of parallel lines cutting off equal segments on one transversal, etc. An assignment for study work requiring the construction of such figures in contrasting inks or crayons does much toward clearing up a demonstration otherwise extremely involved. Incidentally, the use of colored chalk to pick out from a complex figure a certain pair of parallel lines and a transversal or, in solid geometry, to throw into relief a certain plane from a figure involving several planes is a contrivance which will often remove a mass of confusion.

Moreover there is the device of showing a figure with material things, rulers, pencils and compasses, shifting them to various possible positions as one illustrates. The contrivance of covering, for the moment, some part or parts of a figure with the fingers, pencils, pens or rulers, and hence considering the figure in its elements, is most helpful. Youth at fourteen or fifteen has done very little of an analytic nature.

For solid geometry, models constructed of cardboard, glue, wire, butcher pins, hair pins and thread help to visualize figures in three dimensions, during the first several weeks of the study. Daily practice in drawing and understanding these same figures in perspective should of course accompany the use of the models. Models are quite useful too in illustrating problems involving loci, in plane geometry.

If possible, while studying the figure of one theorem, to antici-

pate the next subsequent theorem makes the attack much easier. A discussion and summary of available authorities and ideas having possible bearing upon the proof about to be attempted assists the timid and less venturesome pupil.

Notebooks developed and corrected from day to day containing all of the corollaries of a course are invaluable guides and sources of satisfactory review. A notebook so complete as to include all of the exercises demonstrated by a class is a monumental task. However a notebook of corollaries should present various types of proof and will bring out the possibility of there being often several equally correct demonstrations of the same theorem.

A device helpful in the teaching of converse theorems to put the final emphasis upon the similarity and contrast between any such pair of demonstrations is to require the proof of both theorems, one immediately following the other, using the very same figure and lettering. This is an excellent method of review, after each theorem has been carefully drilled independently. This same plan applies to theorems and their negatives. However, though the same lettering may be employed here, a new construction is sometimes necessary for the negative.

Of very great value are summaries of theorems worked out from time to time to prove certain general truths such as can be reached by several routes, say, the congruence of triangles, or the condition for parallel lines or for the inequality of lines or of angles, or the conditions under which triangles are similar. This sort of collection of available authorities is particularly useful in solving originals. A list of theorems proved by the direct method, another by superposition, another by the indirect method, another list of which there are converses, and another list of which there are negative theorems serve as brief but suggestive review sources.

With respect to an unusually clever class, the presentation of a geometrical fallacy for explanation creates wholesome curiosity, interest and a keen and genuine but fair sense of criticism which carries over for weeks. The fallacy is the peer of puzzles for the teacher of mathematics.

Considering types of proof again, induction, the tool of laboratory sciences and of some other branches of mathematics has little place in formal geometry. This is, in fact, one explanation of the great difficulty of the subject. However this

method should be used whenever possible. It should be employed, for example, in the theorem concerning the sum of the interior angles of a polygon, a proposition otherwise abstruse and, at the best, most difficult and foreign to pupils. This theorem undoubtedly grew out of numbers of specific examples, collected and, finally after quite a period of use, summarized and generalized by a master mind. A tabular derivation should be developed before the class, by experiment in this manner, using in succession

Number of sides of polygon ..	3	4	5	6	7	8	9	... n
Number of $\triangle$ formed by means of all possible diagonals from one vertex	1	2	3	4	5	6	7	... $n-2$
Sum of $\angle$ of the $\triangle$ in right angles..................	2	4	6	8	10	12	14	... $2(n-2)$

a triangle, quadrilateral, etc. It is usually necessary to develop this table more than once so as to render meaning to the generalized theorem and proof. Immediate and numerous applications are necessary in this instance.

But one of the most useful, powerful and practical methods of demonstration in mathematics, elementary or advanced, is often slurred over or neglected entirely in geometry texts and instruction. Every pupil who is devoting as much as two years to the study of mathematics in high school should understand the analytic proof so that he may have it as a tool for general reasoning, if he wishes it. The plan of this method is to assume the conclusion to be true. From this assumption then to proceed step by step by methods of simplifying, justifiable by sound theory, i.e., authorities, until the required theorem is found to depend upon the hypothesis or some other fundamental truth. Then, having the key to the steps themselves and the order of the steps, to reverse the procedure, reasoning from the hypothesis to the conclusion, each step in this, the solution or direct proof, being the converse of the corresponding step of the analysis. This method applies to demonstrations that are largely algebraic as well as to those purely geometrical. It is especially valuable in problems and originals. In a required construction, or problem, by this method the figure is not constructed at first, but merely sketched so as to represent the desired result, then studied

or analyzed so as to determine what parts are to be constructed and the method of so constructing them. In the following example the reasons are omitted:

To construct a square equivalent to a given parallelogram.

Given: Parallelogram $NOPQ$ having base b and altitude a.

Required to construct a square equivalent to the parallelogram $NOPQ$.

Analysis:
1. Let s = side of the required square.
2. Then s^2 = area of the required square.
3. But ab = area of given parallelogram.
4. $\therefore s^2$ must $= ab$.
5. $\therefore a : s = s : b$.
6. $\therefore s$ must be constructed the mean proportional between a and b, and hence can be constructed.

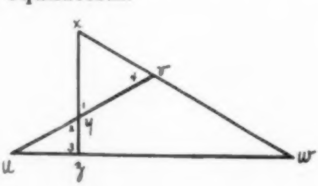
Construction:
1. Construct s the mean proportional between a and b.
2. Using s as a side, construct square S.

Proof:
1. Area of $S = s^2$.
2. Area of parallelogram $NOPQ = ab$.
3. $a : s = s : b$ or $s^2 = ab$.
4. $\therefore$ Area $NOPQ$ = area of S. Q. E. F.

The following is another type of analytic proof. Again the reasons are omitted.

Given: $uv = vw$; $\angle uvw = 4\angle u$; $xz \perp uw$.

To prove: $\triangle xyv$ equilateral.



Analysis

$\triangle xyv$ is equilateral—assuming the conclusion.

$\therefore \quad \angle 1 = 60°$;

$\therefore \quad \angle 2 = 60°$.

But $\angle 3 = 90°$;

$\therefore \quad \angle u = 30°$.

But $\angle w = \angle u$;

$\therefore \quad \angle w = 30°$;

Solution

$\angle uvw = 4\angle u$; $uv = vw$;

$\angle 3 = 90°$ Hypothesis.

$\therefore \quad \angle u = \angle w$.

But $\angle u + \angle w + \angle uvw = 180°$;

$\therefore \quad \angle u + \angle u + 4\angle u = 180°$;

$\therefore \quad \angle u = 30°$; $\angle uvw = 120°$.

But $\angle 2$ is the complement of $\angle u$;

$\therefore \quad \angle uvw = 120°$ or $\angle uvw = 4\angle u$.

$\therefore \quad \angle 2 = 60°$.
But $\angle 1 = \angle 2$;
$\therefore \quad \angle 1 = 60°$.
But $\angle 4$ is the supplement of $\angle uvw$;
$\therefore \quad \angle 4 = 60°$;
$\therefore \quad \angle x = 60°$;
$\therefore \quad \Delta xyv$ is equilateral.
Q. E. D.

The following though a purely algebraic proof is of the type found in Bk. III of plane geometry under the subject of ratio and proportions:

Given: $\dfrac{a}{b} = \dfrac{c}{d}$.

To prove: $\dfrac{ma^2 + nc^2}{mb^2 + nd^2} = \dfrac{a^2}{b^2}$.

Analysis

1. If $\dfrac{ma^2 + nc^2}{mb^2 + nd^2} = \dfrac{a^2}{b^2}$.	1. Assuming conclusion.
2. Then, $ma^2b^2 + nc^2b^2 = ma^2b^2 + na^2d^2$.	2. Theorem of proportions.
3. And $+ ma^2b^2 = + ma^2b^2$.	3. Identity.
4. $\therefore nc^2b^2 = na^2d^2$.	4. Axiom for subtraction.
5. $\therefore b^2c^2 = a^2d^2$.	5. Axiom for division.
6. $\therefore bc = ad$.	6. Axiom for evolution.
7. $\therefore \dfrac{a}{b} = \dfrac{c}{d}$.	7. Theorem of proportions.

Solution

1. $\dfrac{a}{b} = \dfrac{c}{d}$.	1. Hypothesis.
2. $bc = ad$.	2. Theorem of proportions.
3. $b^2c^2 = a^2d^2$.	3. Axiom for involution.
4. $\therefore nb^2c^2 = na^2d^2$.	4. Axiom for multiplication.
5. Also $ma^2b^2 = ma^2b^2$.	5. Identity.
6. $\therefore ma^2b^2 + nb^2c^2 = ma^2b^2 + na^2d^2$.	6. Axiom for addition.
7. $\therefore b^2(ma^2 + nc^2) = a^2(mb^2 + nd^2)$.	7. Factoring.
8. $\therefore \dfrac{ma^2 + nc^2}{mb^2 + nd^2} = \dfrac{a^2}{b^2}$.	8. Theorem of proportions.

Q. E. D.

The use of analysis need not in every instance necessarily be in the form of a rigorous complete double proof as here demonstrated, but may merely serve to bring about suggestions and flashes of ideas which rearrange themselves with some experimenting in a direct proof. In fact we use this method constantly in our daily reasoning everywhere though we can not always trace all of the steps of the two-fold process. We actually rarely take all of the steps.

These then are the types of thinking process we teachers of geometry wish to arouse and develop. These too are some of our stratagems and devices which tend to bestir the human mind naturally inclined to be slothful. No doubt many of these schemes are regularly used by teachers ever alert for the discovery of aids to the technique of teaching this difficult subject. Likely there are teachers who can materially add to such a list of inventions. However there may be some instructors for whom such an exchange of ideas would be helpful.

NEWS AND NOTES OF THE BOSTON MEETING

The program of the National Council of Teachers of Mathematics as printed in the February number of the MATHEMATICS TEACHER was carried out as scheduled. It was estimated that about 200 different people attended the meeting. The Council voted to incorporate and accepted the by-laws as printed in the January number of the TEACHER with certain modifications. Article II, Section 2, was changed so as to permit anyone who paid the annual dues to become a member. Article III, Section 1, was changed so as to read "The Officers of the Council shall be a President, two (2) Vice-Presidents, a board of directors and so on." Article V, Section 1, was changed so as not to impose a minimum limit on the number who might petition the Council for group membership. Section 2 of Article V was changed to Section 3, and a new Section 2 was added as follows: "A branch may and should have a report of all its meetings published in the official journal and shall have the right to send a voting delegate to all the meetings of the Council."

The names of the new officers of the Council appear on the inside front cover page, and they correspond to the new scheme of officers as given in the new by-laws.

Practically all of the papers read at the meeting will appear in this and in subsequent issues of the journal, so no further comment on the program need be made here.

A great deal of interest was manifested in the plan of the Editors of the MATHEMATICS TEACHER to secure a membership of 10,000 by 1930. This can now be accomplished if only each member will secure at least one new member.

The Council also decided to issue a register of members as a supplement to one of the future numbers of the MATHEMATICS TEACHER. This will be published in pamphlet form and will serve as a "Who's Who?" in mathematics. All teachers of mathematics who desire to be represented in this register should send in their dues at once, together with name, address, and teaching position as they wish to have them appear in the register of members.